Math 54: Linear Algebra and Differential Equations

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Gaussian Elimination

Any system can be represented using an **augmented matrix**. When we use elimination to solve a system of linear equations, all information is contained in the coefficients, so we only need to keep track of the coefficients when we solve a system of linear equations.

Example: Here is an augmented matrix for a non-homogeneous system of linear equations:

$$x_1 - 3x_2 = 2$$

$$-2x_1 + x_2 = 3$$

$$\begin{bmatrix} 1 & -3 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

Here is an augmented matrix for a homogeneous system of linear equations:

$$x_1 - 2x_2 + 3x_3 = 0$$

$$3x_1 + 2x_2 - 2x_3 = 0$$

$$2x_1 + 4x_2 - 5x_3 = 0$$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & 2 & -2 & 0 \\ 2 & 4 & -5 & 0 \end{bmatrix}$$

When we solve a system by elimination, we can switch the order of equations, multiply equations by nonzero constants, or add multiples of one equation to another. In matrix form, this means that we can perform any of the following **elementary row operations** without changing the solutions of the system that the augmented matrix represents:

- Change the order of any two rows.
- Multiply a row by a *nonzero* constant.
- Replace a row by that row plus a multiple of another row.

Using these elementary row operations, we can put an augmented matrix into a form called **redcued row-echelon form**. Each augmented matrix has a unique reduced row-echelon form.

A matrix in reduced row-echelon form has the following properties:

- All rows of zeros are below any nonzero rows.
- The first nonzero entry in every nonzero row is 1 (these are called **pivots**) and the pivots move to the right as you go down each nonzero row.
- Any column with a pivot is all zero, except for the pivot (which is 1).

This form lets us read off the solutions of the system easily. Any columns that do not contain a pivot (which we will call a **free column**) correspond to a variable, which we call a **free variable**.

A system is **consistent** (has at least one solution) if there are no rows of the form [0 0 ... 0 b] where b is nonzero in the row-echelon form. A system is **inconsistent** (has no solutions) if there is a row of the form [0 0 ... 0 b] where b is nonzero in the row-echelon form. If a consistent system has at least one free variable, it has infinitely many solutions.

If a solution has infinitely many solutions, we can parametrize the solution space using the free variables. There should be one parameter for every free variable.

Important Warning: It is not true that if a system's reduced row-echelon form has a row of zeros, then the system has infinitely many solutions! As an example, consider the system of equations in 3 equations and 2 variables,

$$x + y = 2$$

$$2x - y = 1$$

$$x - 2y = -1$$

Problem 1

Solve the following systems.

$$3x + y + z = 4$$

$$x - 2y + 4z = -7$$

$$2x - y - 5z = 5$$

$$x + 2y + z = 1$$

$$2x + 5y + 3z = 2$$

$$-x - y = -1$$

$$x + 2y + z = 0$$
$$-2x + 6y = 0$$
$$3x - 2y - z = 0$$

$$x - y - z = 2$$
$$2x - 5y + 3z = 1$$
$$4x - 7y + z = 0$$

$$x + 4y = -2$$
$$-2x - y = -3$$
$$3x + 2y = 4$$

$$x + 3y + z = 2$$
$$2x + 3y - 3z = 1$$

$$x + y - 2z = 1$$
$$2x + 2y - 4z = 2$$
$$-x - y + 2z = -1$$

Problem 2

For what values of h is the following system consistent?

$$x + 2y + hz = 3$$
$$2x + 3y - z = 1$$
$$4x + 7y + 3z = 4$$

Problem 3

Show that an underdetermined system (a system where there are more variables than equations) is always consistent.

Problem 4

Find an example of an inconsistent homogeneous system of equations or show that one cannot exist (in which case every homogeneous system of equations is consistent).

Vectors in $\mathbb{R}^n$

A **vector in** $\mathbb{R}^n$ is a collection of n real numbers,

$$\mathbf{v} = (b_1, b_2, ..., b_n)$$

This can also be written as

$$\mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

We can add vectors in the usual way. For example, if $\mathbf{v} = (2, 1, 1)$ and $\mathbf{w} = (3, 2, -1)$, then

$$\mathbf{v} + 3\mathbf{w} = (11, 7, -2)$$

We can visualize vectors as either points in $\mathbb{R}^n$ or as directed arrows. We can visualize the addition of vectors using the **parallelogram law** and we can visualize multiplying a vector by a real number as a **dilation** (scaling by a certain factor).

The set of **linear combinations** of vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}$ is the set of vectors of the form

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_k\mathbf{v_k}$$

for real numbers c_i.

For example, the vector $(-5, 1)$ is a linear combination of the vectors $(-2, 1)$ and $(-1, 1)$.

$$4(-2, 1) - 3(-1, 1) = (-5, 1)$$

Problem 5

Determine if the vectors $(0, 0, 0)$, $(1, 1, 0)$, and $(1, -6, 5)$ are linear combinations of the two vectors $\mathbf{v_1} = (1, 2, -1)$ and $\mathbf{v_2} = (2, 0, 1)$.