Math 54: Linear Algebra and Differential Equations

Jeffrey Kuan

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1 Review

Recall that a **vector space** over $\mathbb{R}$ is an abstract set of objects called **vectors**, equipped with the operations of vector addition and multiplication by scalars (real numbers) that satisfy given properties (the most important of which are that the vector space is **closed** under vector addition and scalar multiplication).

We then discussed functions that map between vector spaces. Let V and W be vector spaces over $\mathbb{R}$. A function $f : V \to W$ is called a **linear transformation** if

$$f(c_1 u + c_2 v) = c_1 f(u) + c_2 f(v)$$

for all vectors u, v in V and for all real numbers c_1 and c_2.

We then defined the kernel and range of a linear transformation $f : V \to W$.

- Recall that the **kernel** of a linear transformation is the set of vectors v in V such that $f(v) = 0$, where 0 here denotes the zero vector in W.
- The **range** of a linear transformation is the set of vectors w in W such that there is a vector v in V with $f(v) = w$.

Example 1.1. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ given by

$$T(x_1, x_2) = 2x_1 - x_2$$

- The kernel is the set of (x_1, x_2) such that $T(x_1, x_2) = 2x_1 - x_2 = 0$. So we see that the kernel is all vectors of the form $(s, 2s)$ where s is a real number.
- The range is the set of all real numbers that can be obtained as $2x_1 - x_2$ from some (x_1, x_2). But given any real number c, note that $T(0, -c) = c$. So every real number is in the range. So the range is all real numbers.

Example 1.2. Consider the linear transformation, $\text{tr} : M_{3\times 3} \to \mathbb{R}$, given by the trace.

- The kernel is the set of matrices that have zero trace. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$ is in the kernel.
- The range is all real numbers. This is because given any real number c,

$$\text{tr}\left(\begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = c$$

2 One-to-One and Onto

There are two natural questions we can ask about functions. First, does every element get mapped to a unique value (one-to-one)? And second, does the function map onto every element in the set of potential values (onto)? We can consider these notions for a linear transformation.

Definition: Let $f : V \to W$ be a linear transformation between two vector spaces.

- f is **one-to-one** if $f(v_1) = f(v_2)$ implies that $v_1 = v_2$.
- f is **onto** if for every vector w in W, there is v in V such that $f(v) = w$.

These definitions conceptually make sense. But they can be difficult to use. The following conditions are much more useful.

- $f : V \to W$ is **one-to-one** if and only if $\ker(f) = \{0\}$. (The only element in the kernel of f is the zero vector in V.)
- $f : V \to W$ is **onto** if and only if $\text{range}(f) = W$. (Every vector in W is in the range of f.

Example 2.1. The map $T(x_1, x_2) = 2x_1 - x_2$ from $\mathbb{R}^2$ to $\mathbb{R}$ is not one-to-one, but it is onto. The trace $\text{tr} : M_{3\times 3} \to \mathbb{R}$ is not one-to-one, but it is onto.

A linear transformation $f : V \to W$ that is both one-to-one and onto is **bijective**. A bijective linear transformation associates elements in V and elements in W in a nice way, where every vector in V is associated to a vector in W and every vector in W is associated to a vector in V. This means that the linear transformation gives a "translation" between vectors in V and vectors in W.

For a bijective linear transformation $f : V \to W$, we can define the **inverse linear transformation** $f^{-1} : W \to V$, where

$$f^{-1}(w) = v \text{ if and only if } f(v) = w$$

So in essence, f^{-1}, if you give it w in W, tells you which vector v in V gets mapped to it by f. So f^{-1} undoes f. In particular,

$$f^{-1} \circ f = f \circ f^{-1} = \text{Id}$$

where $\text{Id} : V \to V$ is the identity linear transformation.

Problem 1

Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is bijective and find T^{-1}.

Problem 2

Show that the transpose linear transformation $t : M_{2\to2} \to M_{2\to2}$ is bijective and find t^{-1}.

3 Subspaces of Vector Spaces

Recall that we defined a vector space as an abstract set with vectors, vector addition, and scalar multiplication that satisfied the following properties.

- V is closed under addition, so $\mathbf{u} + \mathbf{v}$ is in V, whenever $\mathbf{u}$ and $\mathbf{v}$ are.
- Addition of vectors is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Addition of vectors is associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- There is some zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- Every vector $\mathbf{u}$ has an additive inverse $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$
- For every scalar (real number) c and every vector $\mathbf{u}$, $c\mathbf{u}$ is in V.
- The distributive property for scalar multiplication holds: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

When we define a mathematical structure, we are usually interested in "nested" instances of that structure. *In particular, is it possible for a vector space to have a vector space inside of it?*

Definition: A **subspace** is a subspace of a vector space that is also a vector space.

Note that if you have a subset of a vector space, that subset inherits most of the properties above from the vector space that it is sitting in. In particular, to check that something is a subspace we only need to check the following condition.

Important Result: If U is a subset of a vector space V, then U is a **subspace** of V if for each u_1, u_2 in U, $c_1u_1 + c_2u_2$ is in U too, where c_1 and c_2 are arbitrary real numbers.

Examples of Subspaces:

- Recall that $M_{3\times 3}$, the set of 3 by 3 matrices is a vector space. The subset $\text{Sym}_{3\times 3}$ of symmetric 3 by 3 matrices is a subspace of $M_{3\times 3}$, since for any two symmetric matrices A and B, $c_1A + c_2B$ is still a symmetric matrix.

- Recall that $\mathbb{R}^3$ is a vector space. Then, the set S of vectors $(x_1, x_2, 0)$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, since
$$c_1(a_1, a_2, 0) + c_2(b_1, b_2, 0) = (c_1a_1 + c_2b_1, c_1a_2 + c_2b_2, 0)$$
is still in S (since the last coordinate is still zero).

- Recall that P_n is a vector space. The set of polynomials with constant term equal to zero is a subspace, since $c_1p(x) + c_2q(x)$ has constant term equal to zero if $p(x)$ and $q(x)$ have constant term equal to zero.

- Every vector space is a subspace of itself.

- Let $\{0\}$ be the set of just the zero vector. Then $\{0\}$ is a subspace of any vector space. It is called the **trivial (zero) subspace**.

- The set of vectors in $\mathbb{R}^3$ with first coordinate equal to 1 is not a subspace of $\mathbb{R}^3$, since $2(1, 1, 1) = 2(1, 1, 1) + 0(1, 1, 1) = (2, 2, 2)$ which is a vector that does not have first coordinate equal to 1.