Math 54: Lecture 7/31/19

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More About Symmetric Matrices

As a review from last time, let's begin by orthogonally diagonalizing a symmetric matrix.

Problem 1

Orthogonally diagonalize the symmetric matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

We know that a symmetric matrix has n real eigenvalues counted with multiplicity. It turns out that there is a nice characterization of the largest and smallest eigenvalue of a symmetric matrix A. It is given by the **Rayleigh principle**, which you will prove in Problem Set 9.

Rayleigh principle: Let A be an n by n symmetric matrix. Let λ_{min} and λ_{max} denote the smallest and largest eigenvalue of A. Then,

$$\lambda_{\text{min}} = \min_{v \in \mathbb{R}^n, v \neq 0} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$$

$$\lambda_{\text{max}} = \max_{v \in \mathbb{R}^n, v \neq 0} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$$

Note that the minimum for λ_{min} is attained exactly when v is an eigenvector for λ_{min} and the maximum for λ_{max} is attained exactly when v is an eigenvector for λ_{max}.

Example: As an example, let us use the Rayleigh principle to calculate the eigenvalues of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is clearly a symmetric matrix. We calculate for $v = (x, y)$

$$Av = \begin{bmatrix} x + y \\ x + y \end{bmatrix}$$

$$\langle Av, v\rangle = x^2 + 2xy + y^2 = (x+y)^2$$
$$\langle v, v\rangle = x^2 + y^2$$

So we have that

$$\frac{\langle Av, v\rangle}{\langle v, v\rangle} = \frac{(x+y)^2}{x^2+y^2} = 1 + \frac{2xy}{x^2+y^2}$$

Note that $|2xy| \leq x^2 + y^2$, so $-1 \leq \frac{2xy}{x^2+y^2} \leq 1$. We have $\frac{2xy}{x^2+y^2} = 1$ when $x = y$ and $\frac{2xy}{x^2+y^2} = -1$ when $x = -y$.

So we see that

$$\lambda_{\min} = \min_{v \in \mathbb{R}^n, v \neq 0} \frac{\langle Av, v\rangle}{\langle v, v\rangle} = 0 \quad \text{attained for } v = (x, y) \text{ where } x = -y$$

$$\lambda_{\max} = \max_{v \in \mathbb{R}^n, v \neq 0} \frac{\langle Av, v\rangle}{\langle v, v\rangle} = 2 \quad \text{attained for } v = (x, y) \text{ where } x = y$$

This shows that the eigenvalues of A are $\lambda = 0$ and $\lambda = 2$.

Problem 2

Let n be an *even* positive number. Consider the n by n matrix A where every entry is equal to 1. Show that $\lambda_{\min} \leq 0$ and $\lambda_{\max} \geq n$.

Positive Definite and Positive Semidefinite Matrices

Now that we know the spectral theorem for symmetric matrices and Rayleigh's principle, we can talk about an important class of matrices that are important in applications to machine learning, AI, and data science. These are called positive definite and positive semidefinite matrices.

Definition: A **positive definite matrix** is an n by n symmetric matrix A such that $\langle Av, v\rangle > 0$ for all nonzero vectors $v \in \mathbb{R}^n$. A **positive semidefinite matrix** is an n by n symmetric matrix A such that $\langle Av, v\rangle \geq 0$ for all nonzero vectors v in $\mathbb{R}^n$. (Note that every positive semidefinite matrix is also positive definite.)

Note that by Rayleigh's principle, we have that for a positive definite matrix, $\lambda_{\min} > 0$, since

$$\frac{\langle Av, v\rangle}{\langle v, v\rangle} > 0 \text{ for all nonzero } v \in \mathbb{R}^n$$

and for a positive semidefinite matrix, $\lambda_{\min} \geq 0$, since

$$\frac{\langle Av, v\rangle}{\langle v, v\rangle} \geq 0 \text{ for all nonzero } v \in \mathbb{R}^n$$

So here is another characterization of positive definite and positive semidefinite matrices. *A positive definite matrix is a symmetric matrix where all eigenvalues are positive. A positive semidefinite matrix is a symmetric matrix where all eigenvalues are nonnegative.*

Problem 3

Positive semidefinite matrices have nice properties. For example, show the following property: given a positive semidefinite A, there exists another positive semidefinite matrix B such that $A = B^2$ (so that B is like the square root of A). This is the matrix analogue of the fact that every nonnegative number has a square root.

Problem 4

Determine which of the following matrices are positive definite and positive semidefinite.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Finally, we discuss an easy criterion to determine whether a given matrix is positive definite or not. In principle, we could determine whether a matrix is positive definite simply by calculating its characteristic polynomial, finding all n real roots of this polynomial (which are the eigenvalues), and then checking to see if all eigenvalues are positive. However, this takes a lot of time. There is a much simpler criterion to check whether a symmetric matrix is positive definite, which is called **Sylvester's criterion**. A n by n symmetric matrix is positive definite if and only if the determinant of every j by j submatrix in the upper left corner of the matrix is positive, for each $1 \leq j \leq n$.

For example, we can check that B is positive definite since

$$|2| = 2 > 0 \qquad \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0 \qquad \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 4 > 0$$

so Sylvester's criterion shows that B is positive definite. Meanwhile, A is not positive definite since the determinant of the 1 by 1 upper left submatrix $[0]$ is not positive.

Interestingly, there is no analogue of Sylvester's criterion for positive semidefinite matrices (Problem Set 9). So to check whether a matrix is positive semidefinite, you must actually compute all eigenvalues and check that they are all nonnegative, or check that $\langle Av, v\rangle \geq 0$ for all nonzero $v \in \mathbb{R}^n$.