Math 54: Linear Algebra and Differential Equations

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Abstract Vector Spaces and Linear Transformations

Now, we will start the study of abstract linear algebra. What we have done so far is learn the tools of linear algebra, but remember that algebra is the study of mathematical structure. What mathematical structure will we be studying? We will be studying the mathematical structure known as a **vector space**.

Definition: A **vector space** V **over** $\mathbb{R}$ is any set of objects, which we will call **vectors**, with two operations (1) **addition of vectors** and (2) **multiplication of scalars in** $\mathbb{R}$ that satisfy the following properties (where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ will denote arbitrary vectors in V):

- V is closed under addition, so $\mathbf{u} + \mathbf{v}$ is in V, whenever $\mathbf{u}$ and $\mathbf{v}$ are.
- Addition of vectors is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Addition of vectors is associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- There is some zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- Every vector $\mathbf{u}$ has an additive inverse $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$
- For every scalar (real number) c and every vector $\mathbf{u}$, $c\mathbf{u}$ is in V.
- The distributive property for scalar multiplication holds: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Definition: A **linear transformation** between two vector spaces V and W is a map $T : V \to W$ such that $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$. A **linear operator** on V is a linear transformation from a vector space V to itself.

Examples of Vector Spaces:

- $\mathbb{R}^n$ where $n \geq 1$.
- P_n, the polynomials with real coefficients in x with degree less than or equal to n.
- The set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, $C(\mathbb{R})$.
- The set of periodic functions on the interval $[0, 2\pi]$.
- The set of 3×3 matrices, $M_{3\times3}$.

Examples of Linear Transformations:

- $f : \mathbb{R}^m \to \mathbb{R}^n$, where $f(v) = Av$, where $v \in \mathbb{R}^m$ and A is an $n \times m$ matrix.
- For any vector space, $f : V \to V$, given by $f(v) = 0$, the zero linear transformation.
- For any vector space, $f : V \to V$, given by $f(v) = Iv = v$, the identity linear transformation.
- $\frac{d}{dx} : P_n \to P_{n-1}$, given by the derivative.
- $F : P_n \to P_{n+1}$, given by $F(f) = \int_0^x f(t)dt$ (the indefinite integral with $+C$ being $+0$).
- $F : P_n \to \mathbb{R}$, given by evaluation at a point in $\mathbb{R}$, for example, $F(f) = f(0)$.
- $F : M_{3\times3} \to M_{3\times3}$, given by $F(A) = AB$, where B is a fixed 3×3 matrix.
- $\mathrm{tr} : M_{3\times3} \to \mathbb{R}$, given by the trace of the matrix.
- $F : C(\mathbb{R}) \to C(\mathbb{R})$, given by multiplication by x, $F(f) = xf(x)$.

Note that the set of 2 by 2 matrices with determinant zero is NOT a vector space (not closed under addition). Also, the set of matrices of all sizes is NOT a vector space (no natural notion of addition for matrices of different sizes). The set of invertible 3 by 3 matrices is NOT a vector space either (no additive identity 0, not closed under addition).

Next, we talk about the **kernel** and **range** of a linear transformation. Let $f : V \to W$ be a linear transformation between two vector spaces. The **kernel** of a linear transformation is the set of vectors v in V such that $f(v) = 0$. The **range** of a linear transformation is the set of vectors w in W such that $f(v) = w$ for some vector v. (The range is the set of values that f can take on in W.)

Problem 1

What is the kernel and range of each of the linear transformations given above?