Math 54: Lecture 7/29/19

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The Transpose of an Abstract Linear Transformation

Finally, we will use the Riesz representation theorem to define the transpose of an abstract linear transformation $T : V \to W$, where V and W are finite dimensional inner product spaces (so Euclidean spaces). This is different from the matrix transpose, since we are considering the transpose of an arbitrary abstract linear transformation (which is not necessarily given by a matrix multiplication transformation).

Given any linear transformation $T : V \to W$, we will define a **transpose linear transformation** $T^t : W \to V$ as follows. Note that given any $w \in W$, the map

$$\ell_w(v) = \langle Tv, w \rangle$$

is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ so it is a linear functional. Then, by the Riesz representation theorem,

$$\ell_w(v) = \langle Tv, w \rangle = \langle v, u \rangle$$

for some *unique* vector $u \in V$. We define $u = T^t(w)$ where we can define the transpose this way since for every w, such a vector u exists and is *unique*. So we can define the transpose $T^t : W \to V$ as the unique map satisfying

$$\langle Tv, w \rangle = \langle v, T^t w \rangle$$

for every $v \in V$, $w \in W$.

Problem 1

Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 - x_4)$$

Find $T^t(2, 1)$. Then, find a formula for $T^t(y_1, y_2)$.

Least Squares

Earlier in this class, we considered the problem of finding a solution to the system of equations $A\mathbf{x} = \mathbf{b}$. We called such a system **consistent** if there was at least one solution, and we called such a system **inconsistent** if it did not have a solution. We learned how to find all solutions of a consistent solution, and in the case of an inconsistent solution, we simply said there was no solution.

But what if we wanted to say more in the case of an inconsistent system $A\mathbf{x} = \mathbf{b}$? In this case, there is no vector $\mathbf{x}$ that gives $A\mathbf{x} = \mathbf{b}$, but can we maybe find some vector $\hat{\mathbf{x}}$ that gives the best possible approximation to a solution to $A\mathbf{x} = \mathbf{b}$, an "almost solution" of sorts?

This is the idea behind least squares. Let A be an m by n matrix. We will say that $\hat{\mathbf{x}}$ is a **least squares solution** or least squares best approximation to the system $A\mathbf{x} = \mathbf{b}$ if

$$||A\hat{\mathbf{x}} - \mathbf{b}|| \leq ||A\mathbf{x} - \mathbf{b}|| \text{ for all } \mathbf{x} \in \mathbb{R}^n$$

We call $||A\hat{\mathbf{x}} - \mathbf{b}||$ the least squares distance. Intuitively, this means that $\hat{\mathbf{x}}$ makes $A\hat{\mathbf{x}}$ as close as possible to $\mathbf{b}$, so that even if $A\hat{\mathbf{x}}$ is not equal to $\mathbf{b}$, the least squares solution has $A\hat{\mathbf{x}}$ as close to $\mathbf{b}$ as possible. This shortest distance is exactly the least squares distance.

Here are some important remarks about the least square solution.

- If $A\mathbf{x} = \mathbf{b}$ is a consistent system, then the set of least square solutions is just the set of solutions, since the left hand side in the above inequality can be as small as possible (equal to 0 in this case) precisely when $\hat{\mathbf{x}}$ is a solution to the consistent system.

- Note that there may be many possible least squares approximations to $A\mathbf{x} = \mathbf{b}$. This is precisely the case when the homogeneous $A\mathbf{x} = 0$ has infinitely many solutions, because if $\mathbf{x}_h$ is a solution to the homogeneous equation, then $A\mathbf{x} = A(\mathbf{x} + \mathbf{x}_h)$.

The definition of the least squares solution above says that to find a least square solution, we must minimize the distance between the fixed vector $\mathbf{b}$ and the subspace of all possible $A\mathbf{x}$, which as we already discussed earlier in the class is exactly the **column space of** A. So in particular, we want to minimize the distance from $\mathbf{b}$ to $\mathrm{Col}(A)$. We can do this by using orthogonal projection.

In particular, we should set

$$A\hat{\mathbf{x}} = \mathrm{proj}_{\mathrm{Col}(A)}(\mathbf{b}) \tag{1}$$

Then, the set of all $\hat{x}$ that satisfy the above system is the set of least squares solutions to $A\mathbf{x} = \mathbf{b}$. We will call equation (1) the **associated least squares system for the system** $A\mathbf{x} = \mathbf{b}$. Solving the system of equations (1) gives all least squares solutions.

Note that by the definition of orthogonal projection, this equation also implies that

$$A\hat{\mathbf{x}} - \mathbf{b} \text{ is orthogonal to } \mathrm{Col}(A)$$

In particular, this means the dot product of each column of A with $A\hat{\mathbf{x}} - \mathbf{b}$ is zero, or equivalently, the dot product of each row of A^t with $A\hat{\mathbf{x}} - \mathbf{b}$ is zero. So in particular, $\hat{\mathbf{x}}$ being a least squares solution to $A\mathbf{x} = \mathbf{b}$ is equivalent to

$$A^t(A\hat{\mathbf{x}} - \mathbf{b}) = 0$$

So the set of least squares solutions $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$ is also the solutions to the system

$$A^t A\mathbf{x} = A^t\mathbf{b} \tag{2}$$

The system of equations in (2) is called the **normal system of equations for the system** $A\mathbf{x} = \mathbf{b}$. It is easier to use the normal system rather than the associated system to calculate least squares solutions, since the normal equations involve just matrix multiplication with A^t while the associated system requires calculating an orthogonal projection. However, both would still give the same answer.

As a final note, it is interesting to ask when the least squares solution to $A\mathbf{x} = \mathbf{b}$ is unique. If such a least squares solution is unique, it means that the associated system (1) and the normal system (2) will have unique solutions. Using (2), this means that A^tA is invertible (we can talk about invertibility here since A^tA is a square matrix, though A might not necessarily be a square matrix). Thus, we discover the following important fact: $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution precisely when the square matrix A^tA is invertible.

Problem 2

Find a least squares solution to the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

What is the least squares distance?

Problem 3

Find a least squares solution to the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 2 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

What is the least squares distance?