Math 54: Lecture 7/26/19

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Isometries

Recall that in the previous unit, we considered abstract linear transformations between abstract vector spaces. The notion of equivalence for these was a bijective linear transformation, which identifies the two vector spaces as essentially the "same". For abstract inner product spaces, there will be a general notion of equivalence too.

If $T : V \to W$ is an abstract linear transformation between two inner product spaces V and W, we say that T is an **isometry** (or an **isometric linear transformation** if T is bijective and if T also preserves inner products in the sense that

$$\langle Tv, Tw\rangle_W = \langle v, w\rangle_V \text{ for all } v, w \in V$$

We will specialize to the case where $V = W = \mathbb{R}^n$ for now, but later, when we study Fourier series, we will consider the very interesting case where T is given by the Fourier series map, $V = L^2([0, 2\pi])$ and $W = \ell^2(\mathbb{Z})$.

Problem 1

Show that the identity transformation $I : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry. Then, show that the following maps

$$T_1(x_1, x_2) = (x_2, x_1)$$

$$T_2(x_1, x_2) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

are also isometries of $\mathbb{R}^2$.

Important: Problem 2

Prove that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if and only if it sends an orthonormal basis to another orthonormal basis.

This last result is important because it allows us to characterize all isometries on $\mathbb{R}^n$. In particular, every linear transformation in $\mathbb{R}^n$ is given by a matrix multiplication map by some

n by n matrix A. (This is a variant of the $L(\mathbb{R}^n, \mathbb{R})$ question in Problem Set 6). Recall that in $\mathbb{R}^n$, the standard basis is an orthonormal basis for $\mathbb{R}^n$ and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by multiplication by an n by n matrix A, then the standard basis is sent to the columns of A. (So $T(1, 0, 0, ..., 0)$ = first column of A, $T(0, 1, 0, ..., 0)$ = second column of A), etc.). So by Problem 2 above, T is an isometry when the standard basis is sent to an orthonormal basis, namely when the columns of A form an orthonormal basis for $\mathbb{R}^n$.

So a matrix multiplication map $\mathbb{R}^n \to \mathbb{R}^n$ is an isometry precisely when the columns of A form an orthonormal basis. By using the definition of an orthonormal basis and the definition of matrix multiplication, this means that

$$A^T A = I$$

so that $A^T = A^{-1}$. So $A^T A = AA^T = I$ for an isometric matrix multiplication map. We call a matrix A such that $A^T A = AA^T = I$ an **orthogonal matrix**. *So any isometry in $\mathbb{R}^n$ is given by matrix multiplication by an orthogonal matrix.*

Problem 3

Check that

$$T(x_1, x_2) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

for any fixed α is an isometry of $\mathbb{R}^2$. Describe this isometry geometrically.

The Riesz Representation Theorem

Recall Problem 4 in Problem Set 6, where you found $L(\mathbb{R}^n, \mathbb{R})$, which recall is the set of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}$. A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ is called a **linear functional of $\mathbb{R}^n$**.

We saw in that question that any given linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ is of the form

$$\ell(v) = \langle v, w \rangle$$

for some fixed vector $w \in \mathbb{R}^n$, and conversely, an such map is a linear functional on $\mathbb{R}^n$. This leads to the Riesz representation theorem.

Riesz Representation Theorem: Any linear functional on $\mathbb{R}^n$ is given by a dot product with a unique fixed vector w in $\mathbb{R}^n$.

As a remark, the Riesz Representation Theorem also holds for $L^2([0, 1])$ and $\ell^2(\mathbb{Z})$ with the added assumption that the linear function is a continuous map (but this is beyond the scope of this class).

(Important) Proof: While the proof in Problem Set 6 works, it is not very illuminating. Consider the following abstract proof. Let us first show existence.

Let $\ell : \mathbb{R}^n \to \mathbb{R}$ be any linear functional. If ℓ is the zero functional, we are done since $\ell(v) = 0 = \langle v, 0 \rangle$. So assume ℓ is nonzero. Then $\text{rank}(\ell) = 1$ and the kernel of ℓ, which we denote by $W = \ker(\ell)$ has dimension $n-1$. Let $u_1, u_2, ..., u_{n-1}$ be an orthonormal basis for W. Let u_n be a basis for the one-dimensional space $W^\perp$. Then, $u_1, u_2, ..., u_n$ is an orthonormal basis for $\mathbb{R}^n$. Since $V = W \oplus W^\perp$, the only vector in both $W = \ker(\ell)$ and $W^\perp$ is the zero vector, so u_n is not in $W = \ker(\ell)$.

Let $c = \ell(u_n) \neq 0$ (since u_n is not in the kernel of ℓ). Take $w = cu_n$. Then indeed,

$$\ell(u_n) = \langle u_n, w\rangle = c\langle u_n, u_n\rangle = c$$

as desired, and for $i = 1, 2, ..., n-1$, since $u_n \in \ker(\ell)$, we indeed have

$$\ell(u_n) = \langle u_n, w\rangle = c\langle u_n, u_i\rangle = c \cdot 0 = 0$$

as desired, where we used the orthonormality of $u_1, u_2, ..., u_n$. Since ℓ and the map $T(v) = \langle v, w\rangle$ agree on the orthonormal basis $u_1, u_2, ..., u_n$, they are equal. So $\ell(v) = \langle v, w\rangle$.

For uniqueness, if $\ell(v) = \langle v, w_1\rangle = \langle v, w_2\rangle$, we want to show $w_1 = w_2$. But then, by bilinearity,

$$\langle v, w_1 - w_2\rangle = 0$$

for all vectors v. Taking $v = w_1 - w_2$, we get

$$\langle w_1 - w_2, w_1 - w_2\rangle = 0$$

so by positive definiteness, $w_1 - w_2 = 0$ so $w_1 = w_2$, which shows uniqueness.

Problem 3

Suppose that $\ell : \mathbb{R}^3 \to \mathbb{R}$ is a linear functional with $\ell(1, 1, 0) = 2$, $\ell(1, 3, 0) = -1$, and $\ell(-2, 2, 1) = -3$. Find $\ell(68, 71, 26)$.

The Transpose of an Abstract Linear Transformation

Finally, we will use the Riesz representation theorem to define the transpose of an abstract linear transformation $T : V \to W$, where V and W are finite dimensional inner product spaces (so Euclidean spaces). This is different from the matrix transpose, since we are considering the transpose of an arbitrary abstract linear transformation (which is not necessarily given by a matrix multiplication transformation).

Given any linear transformation $T : V \to W$, we will define a **transpose linear transformation** $T^t : W \to V$ as follows. Note that given any $w \in W$, the map

$$\ell_w(v) = \langle Tv, w\rangle$$

is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ so it is a linear functional. Then, by the Riesz representation theorem,

$$\ell_w(v) = \langle Tv, w\rangle = \langle v, u\rangle$$

for some *unique* vector $u \in V$. We define $u = T^t(w)$ where we can define the transpose this way since for every w, such a vector u exists and is *unique*. So we can define the transpose $T^t : W \to V$ as the unique map satisfying

$$\langle Tv, w \rangle = \langle v, T^t w \rangle$$

for every $v \in V$, $w \in W$.

Problem 4

Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 - x_4)$$

Find $T^t(2, 1)$. Then, find a formula for $T^t(y_1, y_2)$.