Math 54: Linear Algebra and Differential Equations

Jeffrey Kuan

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More on Orthogonality

We will restrict our attention to $\mathbb{R}^n$, or more generally, finite-dimensional inner product spaces. Recall that given an inner product space, we have a notion of perpendicularity or equivalently **orthogonality**, where $\langle v, w \rangle = 0$ if and only if v and w are orthogonal. Recall that we say that a vector v is a **unit vector** if $||v|| = 1$. We use these notions to define the following fundamental concepts.

Definition: Consider any finite set of vectors $v_1, v_2, ..., v_k$ in an inner product space V. We say that the vectors form an **orthogonal set** if $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$ (so every distinct pair of vectors is orthogonal). If in addition, every vector in the set is a unit vector, we say that the vectors form an **orthonormal set**. If in addition, the vectors are a basis for V, the vectors are an **orthonormal basis** for V.

What is good about orthonormal sets? First, it is an important fact that *orthonormal sets are always linearly independent*. To see this, let $v_1, v_2, ..., v_k$ be an orthonormal set. Then, if

$$c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0$$

then

$$\langle c_1 v_1 + c_2 v_2 + ... + c_k v_k, v_1 \rangle = \langle 0, v_1 \rangle = 0$$

which implies by bilinearity and orthonormality that $c_1 = 0$. A similar argument works for all c_i so that all $c_i = 0$. So orthonormal vectors are linearly independent.

Further, there is an additional benefit of having an orthonormal basis instead of just a basis. For a general basis, we have to solve systems of equations to get the coefficients for any linear combination. But for an orthonormal set, we can get the coefficients by taking inner products. For example, if $v_1, v_2, ..., v_n$ are an orthonormal basis for V, then given any vector w, if we want to find c_i such that

$$w = c_1 v_1 + c_2 v_2 + ... + c_n v_n$$

by taking the inner product with c_1 as above, we get that

$$c_1 = \langle w, v_1 \rangle$$

and more generally, that $c_i = \langle w, v_i \rangle$. This makes finding these coefficients much easier!

Given a subspace W of $\mathbb{R}^n$, we can define the **orthogonal complement** of W, denoted $W^\perp$, as the set of all vectors in V that are perpendicular to every vector in W.

Problem 1

Check that $W^\perp$ is a subspace of V. What is $W^\perp$ if $W = 0$? What is $W^\perp$ if $W = V$?

The only vector that is in both W and $W^\perp$ is the zero vector. (Why?) Using this fact, we can show that every vector in W is "linearly independent" from every vector in $W^\perp$. We notate this by saying $V = W \oplus W^\perp$, where this operation is called a direct sum. This means that every vector v in V can be written uniquely as a sum $v = w + u$ where $w \in W$ and $u \in W^\perp$. This also implies that

$$\dim(V) = \dim(W) + \dim(W^\perp)$$

Problem 2

Find the orthogonal complement of $W = \text{span}\{(1, 1, 2, 1), (1, 0, 0, -1)\}$ in $\mathbb{R}^4$.