Math 54: Linear Algebra and Differential Equations

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Determinants

Determinants can also be calculated by using elementary row operations, though each of the elementary row operations has an effect on the determinant.

- Switching rows changes the sign of the determinant.
- Adding a multiple of another row to a given row does NOT change the determinant.
- Multiply a row by a nonzero constant c multiplies the determinant by c.

If you want to simplify a matrix, I would suggest using the row operation of adding a multiple of one row to another, to get the matrix to have an easy row along which you can expand (for example, a row of mostly zeros).

Here is the reason that the determinant is important.

Theorem: The following two statements are equivalent for a square $n \times n$ matrix A.

- The determinant of A is nonzero.
- A is invertible.
- The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
- The homogeneous system $A\mathbf{x} = 0$ only has the trivial solution
- The row reduced echelon form of A is the identity matrix.
- The columns of A are linearly independent.
- The columns of A span $\mathbb{R}^n$.

Note that for these equivalences to hold, we must have that A is a square matrix. Which of these equivalences hold for general (non-square matrices A)?

Finally, we define the trace of a square matrix. The **trace** of a matrix is the sum of the entries along the diagonal. The determinant and the trace satisfy the following important properties.

$$\det(AB) = \det(BA) = \det(A)\det(B)$$

$$\det(A) = \det(A^t)$$

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \qquad \text{(but this is not equal to } \mathrm{tr}(A)\mathrm{tr}(B)\text{)}$$

$$\mathrm{tr}(A) = \mathrm{tr}(A^t)$$

Problem 5

Show that a matrix that has two identical rows has determinant equal to zero.

Problem 6

Calculate the determinant of an upper triangular matrix. Use this to characterize all invertible upper triangular matrices.

Problem 7

Let A be an invertible matrix and let B be a matrix of the same size with $\det(B) = 1$. Find the determinant and trace of $A^{-1}BA$. What does the determinant tell you about the invertibility of $A^{-1}BA$? What is its inverse? What is $(A^{-1}BA)^3$?

Abstract Vector Spaces and Linear Transformations

Now, we will start the study of abstract linear algebra. What we have done so far is learn the tools of linear algebra, but remember that algebra is the study of mathematical structure. What mathematical structure will we be studying? We will be studying the mathematical structure known as a **vector space**.

Definition: A **vector space** V **over** $\mathbb{R}$ is any set of objects, which we will call **vectors**, with two operations (1) **addition of vectors** and (2) **multiplication of scalars in** $\mathbb{R}$ that satisfy the following properties (where $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ will denote arbitrary vectors in V):

- V is closed under addition, so $\mathbf{u} + \mathbf{v}$ is in V, whenever $\mathbf{u}$ and $\mathbf{v}$ are.
- Addition of vectors is commutative: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- Addition of vectors is associative: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- There is some zero vector $\mathbf{0}$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- Every vector $\mathbf{u}$ has an additive inverse $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$
- For every scalar (real number) c and every vector $\mathbf{u}$, $c\mathbf{u}$ is in V.
- The distributive property for scalar multiplication holds: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- $c(d\mathbf{u}) = (cd)\mathbf{u}$
- $1\mathbf{u} = \mathbf{u}$

Definition: A **linear transformation** between two vector spaces V and W is a map $T : V \to W$ such that $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$. A **linear operator** on V is a linear transformation from a vector space V to itself.

Examples of Vector Spaces:

- $\mathbb{R}^n$ where $n \geq 1$.
- P_n, the polynomials with real coefficients in x with degree less than or equal to n.
- The set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, $C(\mathbb{R})$.
- The set of periodic functions on the interval $[0, 2\pi]$.
- The set of 3×3 matrices, $M_{3\times 3}$.

Examples of Linear Transformations:

- $f : \mathbb{R}^m \to \mathbb{R}^n$, where $f(v) = Av$, where $v \in \mathbb{R}^m$ and A is an $n \times m$ matrix.
- For any vector space, $f : V \to V$, given by $f(v) = 0$, the zero linear transformation.
- For any vector space, $f : V \to V$, given by $f(v) = Iv = v$, the identity linear transformation.
- $\frac{d}{dx} : P_n \to P_{n-1}$, given by the derivative.
- $F : P_n \to P_{n+1}$, given by $F(f) = \int_0^x f(t)dt$ (the indefinite integral with $+C$ being $+0$).
- $F : P_n \to \mathbb{R}$, given by evaluation at a point in $\mathbb{R}$, for example, $F(f) = f(0)$.
- $F : M_{3\times 3} \to M_{3\times 3}$, given by $F(A) = AB$, where B is a fixed 3×3 matrix.
- $\mathrm{tr} : M_{3\times 3} \to \mathbb{R}$, given by the trace of the matrix.
- $F : C(\mathbb{R}) \to C(\mathbb{R})$, given by multiplication by x, $F(f) = xf(x)$.