Math 54: Linear Algebra and Differential Equations

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Continuation from Yesterday

Problem 1

Find the determinant, trace, eigenvalues, and eigenvectors of the linear transformation

$$\frac{d}{dx} : P_n \to P_n$$

Problem 2

(From the UC Berkeley Ph.D Preliminary Examination, Fall 1981)

Let $M_{2\times 2}$ be the vector space of all real 2×2 matrices. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$$

and define a linear transformation $L : M_{2\times 2} \to M_{2\times 2}$ by $L(X) = AXB$. Compute the trace and determinant of L.

Review for Midterm 2

Problem 1

Let V be the subspace of continuous functions such that $f(x) = 0$ for $x \geq 1$. V is a vector space over $\mathbb{R}$.

- Draw a picture of a nonzero element of V.
- Define operations of vector addition and scalar multiplication on V. Check that V is closed under vector addition and scalar multiplication. What is the zero vector in V?
- Is $T : V \to V$ defined by $T(f) = xf(x)$ a linear transformation from V to V?

Problem 2

For each part, determine if U is a subspace of V.

- U is the set of matrices in $M_{4\times 3}$ with column space equal to $\mathbb{R}^3$, $V = M_{4\times 3}$
- U is the set of polynomials $p(x)$ in P_{10} with $p'(1) = 0$ and $p(2) = 0$. $V = P_{10}$.
- U is the set of polynomials $p(x)$ in P_{10} with $(\frac{p''(x)}{6})^3 = p(x)$, $V = P_{10}$

Problem 3

Determine (with proof) if the following maps are linear transformations. If so, find the kernel and range of the following linear transformations. If the linear transformation is bijective, find its inverse linear transformation.

- The map $T : \mathbb{C} \to \mathbb{R}^2$ given by $T(a + bi) = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$.
- The map $F : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ given by $F(f) = (f'(x))^2$.
- The map $T : M_{3\times 2}$ to $\mathbb{R}^3$ given by $T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+d \\ b+e \\ c+f \end{bmatrix}$.

Problem 4

Let S be the subspace of polynomials in P_{10} with constant term zero and whose coefficients all sum to 0. Find $\dim(S)$. (Hint: Define a linear transformation $T : P_{10} \to \mathbb{R}^2$ by $T(f) = (E_0(f), E_1(f))$ where E_0 is evaluation at $x = 0$ and E_1 is evaluation at $x = 1$).

Problem 5

Consider the set S of 2×3 matrices of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

where $a + e = 0$, $b + f = 0$, $b + d = 0$ and $c + e = 0$. Show that S is a subspace of $M_{2\times 3}$ and find a basis for S. Prove that the set you find is a basis.

Problem 6

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1, 0, 1) = (2, 1, 0)$$
$$T(1, 1, 0) = (0, 1, 2)$$
$$T(0, 1, 0) = (a, 1, 1)$$

For what values of a is T one-to-one? (Hint: $\{(1, 0, 1), (1, 1, 0), (0, 1, 0)\}$ is a basis for $\mathbb{R}^3$. If you want to use this fact, you must prove it.)

Problem 7

Define the linear transformation $T : P_3 \to P_3$, given by $T(p(x)) = p''(x) - 2p(x)$. Find a basis for $\ker(T)$. What is $\dim(\mathrm{range}(T))$?