Math 54: Linear Algebra and Differential Equations

Jeffrey Kuan

July 18, 2019

Diagonalization

Recall that to find a basis $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ for which a matrix transformation defined by an n by n matrix is diagonal (with respect to $\mathcal{B}$), we need to find n distinct eigenvectors that form a basis for $\mathbb{R}^n$. So in particular, we need n total dimensions of eigenvectors to diagonalize a matrix, meaning that we need the **geometric multiplicity of all eigenvalues to sum to n.**

Suppose that the sum of the geometric multiplicities of the eigenvalues is n. How can we form a basis $\mathcal{B}$ with respect to which A is diagonal? It is an important fact that **eigenvectors for distinct eigenvalues are linearly independent** (Problem Set 7). So, if the sum of the geometric multiplicities is n, we can form a basis $\mathcal{B}$ by simply taking a basis for each eigenspace, and putting them all together in one basis. With respect to this basis then, the linear transformation of matrix multiplication by A is a diagonal matrix.

Note that for a matrix multiplication transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by a square matrix A, $T(v) = Av$, it turns out that A is actually the matrix of T with respect to the standard basis. So if D is the diagonal matrix $[T]_{\mathcal{B}\to\mathcal{B}}$ with respect to the basis of eigenvectors, then we have that

$$A = [T]_{\text{std}\to\text{std}} = [I]_{\mathcal{B}\to\text{std}}[T]_{\mathcal{B}\to\mathcal{B}}[I]_{\text{std}\to\mathcal{B}} = PDP^{-1}$$

where P is the change of basis matrix from $\mathcal{B}$ (the basis of eigenvectors) to the standard basis, and D is the diagonal matrix of T with respect to the basis $\mathcal{B}$ of eigenvectors.

0.1 Problem 1

Diagonalize the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

So when is a matrix not diagonalizable? There are two ways that this can happen.

The first obstacle to diagonalization is that the scalars we are using (real numbers) are not algebraically closed. To illustrate the first case where a matrix is not diagonalizable, consider the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This matrix is not diagonalizable, because its characteristic polynomial is $\text{char}_A(x) = x^2 + 1$, which has no real roots. So this matrix has no eigenvalues. So in particular, the characteristic polynomial does not have n roots (counted with multiplicity), meaning that the sum of the algebraic multiplicities is strictly less than n. Since geometric multiplicity is less than or equal to algebraic multiplicity, this means that the geometric multiplicity in this case is less than n so that the matrix is diagonalizable. This obstacle to diagonalization is due to the fact that the real numbers are not **algebraically closed**. In particular, a degree n polynomial with real coefficients does not always have n real roots.

The second obstacle to diagonalization has to do with an actual defect with the matrix itself. In this case, the characteristic polynomial does have n roots (counted with multiplicity) so that the algebraic multiplicities of the eigenvalues sum to n, but then the geometric multiplicities are still less than n. An example of this is given by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which has an eigenvalue 1 with algebraic multiplicity 4 and geometric multiplicity 1.

The last thing we should talk about is the idea of similar matrices. Two square matrices A and B (of the same size) are **similar** if there is an invertible matrix P such that

$$A = PBP^{-1}$$

Note that every invertible matrix represents a change of basis, and vice versa. So similar matrices in some sense are the same linear transformation, just written with respect to different bases.

An easy computation shows that if A and B are similar matrices, then they have the same characteristic polynomial, since

$$\begin{aligned} \text{char}_A(x) = \det(PBP^{-1} - xI) &= \det(PBP^{-1} - xPIP^{-1}) = \det(P(B - xI)P^{-1}) \\ &= \det(P)\det(B - xI)\det(P^{-1}) = \det(B - xI) = \text{char}_B(x) \end{aligned}$$

In addition, similar matrices will have the same eigenvalues with the same geometric and algebraic multiplicities (but the eigenvectors will be different, in particular the eigenvectors of A will be the eigenvectors of B multiplied on the left by P). In particular, this convinces us that it does not matter which basis we use when we compute eigenvalues. So in some sense, eigenvalues and eigenvectors are independent of the coordinates we use and hence are general properties of arbitrary linear transformations.

If we want to compute the determinant, trace, eigenvalues, or eigenvectors of a general abstract linear transformation, this computation says that we can just compute the matrix of the linear transformation with respect to any basis and then do the computations in that basis. **In particular, we can define the determinant and trace of an abstract linear transformation to be the product and sum of the eigenvalues counted with algebraic multiplicity**, since eigenvalues and the characteristic polynomial of a transformation are invariant under choice of coordinates.

Problem 1

Find the determinant, trace, eigenvalues, and eigenvectors of the linear transformation

$$\frac{d}{dx} : P_n \to P_n$$

Problem 2

(From the UC Berkeley Ph.D Preliminary Examination, Fall 1981)

Let $M_{2\times 2}$ be the vector space of all real 2×2 matrices. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$$

and define a linear transformation $L : M_{2\times 2} \to M_{2\times 2}$ by $L(X) = AXB$. Compute the trace and determinant of L.