Math 54: Linear Algebra and Differential Equations

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Eigenvalues and Eigenvectors

Suppose that $T : V \to V$ is an arbitrary linear operator (this is a linear operator since we are starting and ending in the same vector space). Given a basis $\mathcal{B}$ for V, we get a matrix $[T]_{\mathcal{B}\to\mathcal{B}}$ for the linear transformation T with respect to the basis $\mathcal{B}$. For each such ordered basis $\mathcal{B}$, we get out a different matrix $[T]_{\mathcal{B}\to\mathcal{B}}$.

We could imagine asking the following question: Can we find a basis $\mathcal{B}$ for which the form of $[T]_{\mathcal{B}\to\mathcal{B}}$ is as simple as possible - in particular, so that this matrix is diagonal? If this matrix is diagonal, then if the (i, i) entry is λ_i, this is saying that $T(v_i) = \lambda v_i$ where v_i is the ith basis vector in $\mathcal{B}$. So if we want to try to find a basis in which a linear transformation is diagonal, we want to search for nonzero vectors where T applied to that vector gives a multiple of the original vector.

This motivates the following definition.

Definition: Let $T : V \to V$ be a linear operator. A nonzero vector v is called an **eigenvector with eigenvalue** λ if $T(v) = \lambda v$.

As an example, note that $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix transformation defined by the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (sending $v \in \mathbb{R}^2$ to $Av \in \mathbb{R}^2$) since

$$Au = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3u$$

so that u is an eigenvector of the matrix multiplication linear transformation defined by A, with eigenvalue 3.

As another example, given the identity transformation $I : V \to V$ that sends every vector to itself, every nonzero element of V is an eigenvector of I with eigenvalue 1. Given the zero transformation $Z : V \to V$ that sends every vector in V to the zero vector, every nonzero element of V is an eigenvector of A with eigenvalue 0.

Finally, let $C^\infty(\mathbb{R})$ denote all smooth functions on the real line (functions that have infinitely many derivatives). Then, $f(x) = e^{\lambda x}$ is an eigenvector with eigenvalue λ, since

$$\frac{d}{dx}(e^{\lambda x}) = \lambda e^{\lambda x}$$

We will now restrict to the specific case of linear operators which are matrix transformations, which recall are linear transformations from a vector space to itself. For a matrix transformation to map from $\mathbb{R}^n$ to $\mathbb{R}^n$, we must restrict to square matrices. So the problem is, given a square matrix A, can we find all eigenvectors of A?

If A has an eigenvector v, then note that $Av = \lambda v$. Rewriting this in a more convenient way, we have that

$$Av - \lambda v = (A - \lambda I)v = 0$$

So v is an eigenvector of A with eigenvalue λ if and only if v is in the nullspace of $A - \lambda I$. This gives us a nice way of finding all eigenvectors.

If we recall the invertible matrix theorem, this is saying that λ is an eigenvalue of A if and only if $A - \lambda I$ has nontrivial (nonzero) nullspace if and only if $\det(A - \lambda I) = 0$. So to search for eigenvectors, we need to find all λ such that

$$\det(A - \lambda I) = 0$$

and for each such λ, we can find all eigenvectors with that eigenvalue by finding the nullspace of $A - \lambda I$. Note in particular that if we denote by U_λ the set of all eigenvectors of A with eigenvalue λ, with the zero vector added to this set, then U_λ is a subspace of $\mathbb{R}^n$.

It is a fact that eigenvectors for distinct eigenvalues are linearly independent (a fact which can be proved with the Vandermonde matrix). So if we want to find a basis $\mathcal{B}$ for which a matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by an n by n matrix A is diagonal, we need to find n distinct linearly independent eigenvectors (so for each eigenspace, find a basis, and put all the bases together and see if you get n vectors). If there is a basis of n distinct linearly independent eigenvectors for an n by n matrix A, then we say that A and its associated matrix multiplication transformation is **diagonalizable**.

To find all eigenvalues λ, which are all λ such that $\det(A - \lambda I) = 0$, we need to solve the equation $\det(A - xI) = 0$ for x. The polynomial

$$\text{char}_A(x) = \det(A - xI)$$

is an important polynomial associated to the square matrix A, called the **characteristic polynomial**.

The set of eigenvalues as we have discussed is the set of roots of the characteristic polynomial $\text{char}_A(x)$. However, roots of polynomials may have multiplicity. We define the **algebraic multiplicity of an eigenvalue** λ to be its multiplicity as a root of the characteristic polynomial. The **geometric multiplicity of an eigenvalue** λ is the dimension of U_λ (the nullspace of $A - \lambda I$). It is a general fact that **geometric multiplicity of an eigenvalue is always less than the algebraic multiplicity of an eigenvalue**.

Another important fact is that the trace of a matrix is the sum of the roots of the characteristic polynomial (so it is the sum of the eigenvalues repeated with their algebraic multiplicity) and the determinant of a matrix is the product of the roots of the characteristic polynomial (so it is the product of the eigenvalues repeated with their algebraic multiplicity).

Problem 1

Find all eigenvalues and eigenvectors of the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 0 & 2 \\ 1 & 2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$