Math 54: Linear Algebra and Differential Equations

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Change of Basis Matrix

Let V be an abstract vector space. Then, recall that this vector space has many different bases, where no particular basis is canonical. If there are two bases $\mathcal{B}_1$ and $\mathcal{B}_2$, we could write any vector in V in the coordinates of $\mathcal{B}_1$ or in the coordinates of $\mathcal{B}_2$. We will get different numbers in each case, but both sets of coordinates will still describe the same vector.

Many times, it may be useful to switch between the coordinates given by one basis $\mathcal{B}_1$ and the coordinates given by another basis $\mathcal{B}_2$. In particular, if we want a formula to convert between coordinates with respect to $\mathcal{B}_1$ and coordinates with respect to $\mathcal{B}_2$, we can construct what is called a **change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$**.

We will denote the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ by $[I]_{\mathcal{B}_1 \to \mathcal{B}_2}$. The reason for this is because the change of basis matrix, using the terminology from yesterday is just the matrix of the identity linear transformation from V to V with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$, where the identity linear transformation from V to V just sends every vector to itself. Then, we have that

$$[v]_{\mathcal{B}_2} = [I]_{\mathcal{B}_1 \to \mathcal{B}_2}[v]_{\mathcal{B}_1}$$

Problem 1

Let $\mathcal{B}_1 = \{1, 1 - 2x, 3 + 2x + x^2\}$ and let $\mathcal{B}_2 = \{1, x, x^2\}$. Find the change of basis matrices from $\mathcal{B}_1$ to $\mathcal{B}_2$ and from $\mathcal{B}_2$ to $\mathcal{B}_1$.

Problem 2

Consider the natural basis $\mathcal{B}_1$ for the subspace of symmetric 2 by 2 matrices in $M_{2\times 2}$ and consider the basis

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right\}$$

Find the change of basis matrices from $\mathcal{B}_1$ to $\mathcal{B}_2$ and from $\mathcal{B}_2$ to $\mathcal{B}_1$.