Math 54: Linear Algebra and Differential Equations

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The Matrix of a Linear Transformation

Let's motivate the idea of the matrix of a linear transformation with the following example. Suppose that $T : V \to W$ is a linear transformation, where for concreteness, $\dim(V) = 3$ and $\dim(W) = 4$. Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be an ordered basis for V and let $\mathcal{C} = \{w_1, w_2, w_3, w_4\}$ be an ordered basis for W. Suppose that you know that

$$T(v_1) = 2w_1 + 3w_2 + w_3 - w_4$$

$$T(v_2) = -2w_1 + w_2 - w_3 - 2w_4$$

$$T(v_3) = w_1 + w_2 + 2w_3 + 3w_4$$

Then, what is T? Well, since $\{v_1, v_2, v_3\}$ is a basis, we can write any vector in V in the form

$$v = a_1v_1 + a_2v_2 + a_3v_3$$

So then, applying linearity, we get an explicit formula for T applied to every vector v!

$$T(a_1v_1+a_2v_2+a_3v_3) = (2a_1-2a_2+a_3)w_1+(3a_1+a_2+a_3)w_2+(a_1-a_2+2a_3)w_3+(-a_1-2a_2+3a_3)w_4$$

Writing this in coordinates, where we use the basis $\mathcal{B}$ for V and the basis $\mathcal{C}$ for W, we get that

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{B}}\right) = \begin{bmatrix} 2a_1 - 2a_2 + a_3 \\ 3a_1 + a_2 + a_3 \\ a_1 - a_2 + 2a_3 \\ -a_1 - 2a_2 + 3a_3 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} a_1 a_2 a_3 \end{bmatrix}_{\mathcal{B}}$$

This is amazing! Why? Because even though we have an abstract linear transformation, in some sense, our abstract linear transformation is given by a matrix transformation once we fix bases for V and W! The matrix

$$[T]_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -2 & 3 \end{bmatrix}$$

is called **the matrix of the linear transformation T with respect to the ordered bases $\mathcal{B}$ and $\mathcal{C}$.**

In general, given a linear transformation $T : V \to W$, once you fix a basis $\mathcal{B} = \{v_1, v_2, ..., v_m\}$ for V and $\mathcal{C} = \{w_1, w_2, ..., w_n\}$ for V and W respectively, the matrix of the linear transformation T with respect to the ordered bases $\mathcal{B}$ and $\mathcal{C}$ is a matrix where the jth column is given by the coordinates of $T(v_j) \in W$ with respect to the basis $\mathcal{C}$ for W.

Problem 1

Find the matrix of the linear transformation $\frac{d}{dx} : P_3 \to P_2$ with respect to the bases $\mathcal{B}_1 = \{1, x, x^2, x^3\}$ and $\mathcal{C}_1 = \{1, x, x^2\}$. Then find the matrix of the same linear transformation but now with respect to the bases $\mathcal{B}_1 = \{1 + x, 3 - 2x, x^2 + 1, x^3 - x^2 - x - 1\}$ and $\mathcal{C}_1 = \{1, x, x^2\}$. Use the matrix representation to calculate the kernel and range of this map.

Problem 2

The complex numbers $\mathbb{C}$ are given by $a + bi$ where a and b are real numbers, and $i = \sqrt{-1}$.

- Check that $\mathbb{C}$ is a vector space over $\mathbb{R}$. What is its dimension? Note that $\mathbb{R}$ is a subspace of $\mathbb{C}$.
- Consider the conjugation map $c : \mathbb{C} \to \mathbb{C}$ that sends $a + bi$ to $a - bi$. Check that this is a linear transformation. Find the matrix of c, $[c]_{\mathcal{B}_1 \to \mathcal{B}_1}$ with respect to the natural basis $\mathcal{B}_1$ for $\mathbb{C}$.
- Check that $\mathcal{B}_2 = \{2 + 3i, 1 - 2i\}$ is also a basis for $\mathcal{C}$. Find $[c]_{\mathcal{B}_2 \to \mathcal{B}_2}$, $[c]_{\mathcal{B}_1 \to \mathcal{B}_2}$, and $[c]_{\mathcal{B}_2 \to \mathcal{B}_1}$.
- Abstractly determine if c is bijective, and if it is, find its inverse. What does this mean in terms of the matrices above?