# Math 54: Linear Algebra and Differential Equations

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July 12, 2019

## Applications of Rank-Nullity Theorem

So far, our examples have just been verifying the Rank-Nullity Theorem. Let us actually see why it is useful.

## Problem 1

Let $S$ be the subspace of trace-free matrices in $M_{n \times n}$ (matrices with trace equal to 0). What is the dimension of $S$?

Problem 1 demonstrates the following important result. *A nonzero linear functional $f$ on a finite dimensional vector space $V$ (linear transformation from $V$ to $\mathbb{R}$) has $ker(f) = dim(V) - 1$.* (See Problem Set 6)

## Problem 2

Show that there is no one-to-one linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. Show that there is no onto linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$.

Problem 2 demonstrates the following important result. *A linear transformation from a vector space of larger dimension to a vector space of smaller dimension cannot be one-to-one. A linear transformation from a vector space of smaller dimension to a vector space of larger dimension cannot be onto.* (The case of when the vector spaces both have the same size is covered on Problem Set 6.)

Finally, let's consider the rank and nullity of a **matrix transformation**. Recall that a **matrix transformation** is a linear transformation between Euclidean spaces that is defined by matrix multiplication. In particular, if $A$ is an $m$ by $n$ matrix, then we can define the linear transformation

$$T : \mathbb{R}^n \to \mathbb{R}^m \quad T(\mathbf{x}) = A\mathbf{x}$$

For matrix transformations:

- The **kernel of** $T$ is called the **nullspace of** $A$, and it is the solutions to the homogeneous system $A\mathbf{x} = 0$. Writing the solution to the homogeneous system in parametric vector form immediately gives a basis for the kernel of $T$ (equivalently the nullspace of $A$) and hence nullity($T$).

- The **range of** $T$ is called the **column space of** $T$, because it is the span of the columns of the matrix $A$. Putting the columns into the row of a matrix (transpose) and row reducing to reduced row echelon form immediately gives a basis for the column space of $T$ (and hence $\text{rank}(T)$), which will be the nonzero rows of the reduced row echelon form matrix.

### Problem 3

To illustrate the concepts above, find $\text{rank}(T) = \text{nullspace}(A)$ and $\ker(T) = \text{column space}(A)$ for the matrix transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 4 \\ 4 & 1 & 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

where we are denoting $A$ to be the 3 by 5 matrix above.

## Coordinates for Vector Spaces

Consider an abstract vector space $V$ with dimension $n$. Fix an ordered basis $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ for $V$. Recall that by the spanning and linear independence properties of basis, this means that every vector $v$ in $V$ can be expressed *uniquely* as a linear combination of the basis vectors,

$$v = a_1v_1 + a_2v_2 + ... + a_nv_n$$

Suppose that we have another vector $w$ whose unique linear combination is

$$w = b_1v_1 + b_2v_2 + ... + b_nv_n$$

Then, note that

$$v + w = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + ... + (a_n + b_n)v_n \quad cv = (ca_1)v_1 + (ca_2)v_2 + ... + (ca_n)v_n$$

Notice that in essence, all vector operations can be done on the coefficients on each of the basis vectors. So if we fix a particular basis, $\mathcal{B} = \{v_1, v_2, ..., v_n\}$, we can just deal with the coefficients on the basis vectors in the linear combination and do our computations there. In particular, we can write the **coordinates with respect to** $\mathcal{B}$ for $v$ and $w$ as

$$v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{\mathcal{B}} \qquad w = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{\mathcal{B}}$$

**Note that these coordinates only make sense if we know which basis we are writing the coordinates with respect to, since any given abstract vector space has many different bases.** In particular, vector spaces do not have preferred bases.

## Problem 4

Define the two bases $\mathcal{B}_1 = \{1, x, x^2\}$ and $\mathcal{B}_2 = \{1 - x + 2x^2, 1 + x + 3x^2, 2x + x^2\}$ for $P_2$. Calculate the coordinates of $x^2 + x$ in both bases.

Why are coordinates useful? Well, in some sense, they tell us that every $n$ dimensional vector space is basically like $\mathbb{R}^n$, once we fix a basis. Of course, the identification with $\mathbb{R}^n$ is not **canonical**, meaning that there is no preferred basis in which we should interpret any given vector space. But once we agree on a basis and fix it, we can think of the vector space as just $\mathbb{R}^n$.

# The Matrix of a Linear Transformation

Let's motivate the idea of the matrix of a linear transformation with the following example. Suppose that $T : V \to W$ is a linear transformation, where for concreteness, $\dim(V) = 3$ and $\dim(W) = 4$. Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be an ordered basis for $V$ and let $\mathcal{C} = \{w_1, w_2, w_3, w_4\}$ be an ordered basis for $W$. Suppose that you know that

$$T(v_1) = 2w_1 + 3w_2 + w_3 - w_4$$

$$T(v_2) = -2w_1 + w_2 - w_3 - 2w_4$$

$$T(v_3) = w_1 + w_2 + 2w_3 + 3w_4$$

Then, what is $T$? Well, since $\{v_1, v_2, v_3\}$ is a basis, we can write any vector in $V$ in the form

$$v = a_1v_1 + a_2v_2 + a_3v_3$$

So then, applying linearity, we get an explicit formula for $T$ applied to every vector $v$!

$$T(a_1v_1+a_2v_2+a_3v_3) = (2a_1-2a_2+a_3)w_1+(3a_1+a_2+a_3)w_2+(a_1-a_2+2a_3)w_3+(-a_1-2a_2+3a_3)w_4$$

Writing this in coordinates, where we use the basis $\mathcal{B}$ for $V$ and the basis $\mathcal{C}$ for $W$, we get that

$$T\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{\mathcal{B}}\right) = \begin{bmatrix} 2a_1 - 2a_2 + a_3 \\ 3a_1 + a_2 + a_3 \\ a_1 - a_2 + 2a_3 \\ -a_1 - 2a_2 + 3a_3 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} a_1 a_2 a_3 \end{bmatrix}_{\mathcal{B}}$$

**This is amazing!** Why? Because even though we have an abstract linear transformation, in some sense, our abstract linear transformation is given by a matrix transformation once we fix bases for $V$ and $W$! The matrix

$$[T]_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -2 & 3 \end{bmatrix}$$

is called **the matrix of the linear transformation $T$ with respect to the ordered bases $\mathcal{B}$ and $\mathcal{C}$.**

In general, given a linear transformation $T : V \to W$, once you fix a basis $\mathcal{B} = \{v_1, v_2, ..., v_m\}$ for $V$ and $\mathcal{C} = \{w_1, w_2, ..., w_n\}$ for $V$ and $W$ respectively, the matrix of the linear transformation $T$ with respect to the ordered bases $\mathcal{B}$ and $\mathcal{C}$ is a matrix where the $j$th column is given by the coordinates of $T(v_j) \in W$ with respect to the basis $\mathcal{C}$ for $W$.

# Problem 5

Find the matrix of the linear transformation $\frac{d}{dx} : P_3 \to P_2$ with respect to the bases $\mathcal{B}_1 = \{1, x, x^2, x^3\}$ and $\mathcal{C}_1 = \{1, x, x^2\}$. Then find the matrix of the same linear transformation but now with respect to the bases $\mathcal{B}_1 = \{1 + x, 3 - 2x, x^2 + 1, x^3 - x^2 - x - 1\}$ and $\mathcal{C}_1 = \{1, x, x^2\}$.