Math 54: Linear Algebra and Differential Equations

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1 Basis and Dimension

Let's think about $\mathbb{R}^n$ again. An easy set of vectors to think about in $\mathbb{R}^3$ for example is the set of vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Note that this set of vectors is **both** linearly independent and spans $\mathbb{R}^3$. So we say that this set is a **basis** for $\mathbb{R}^3$.

What is so useful about this particular basis for $\mathbb{R}^3$, known as the **standard basis**? Note that since these vectors span $\mathbb{R}^3$, every vector in $\mathbb{R}^3$ can be written as a linear combination of $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ and linear independence tells us that this linear combination is unique. For example,

$$(2,-3,4) = 2(1,0,0) + (-3)(0,1,0) + 4(0,0,1)$$

and this is the unique way of expressing $(2,-3,4)$ as a linear combination of the basis vectors. The fact that there are three linearly independent vectors in $\mathbb{R}^3$ that span $\mathbb{R}^3$ tells us that in some sense, $\mathbb{R}^3$ only has three distinct directions, so it makes sense to say that the dimension of $\mathbb{R}^3$ is 3, the number of elements in the basis.

We now extend this definition to arbitrary vector spaces in the following way.

Definition: Let V be an arbitrary vector space. An **ordered basis** $\mathcal{B}$ is an ordered list of vectors $\{v_1, v_2, ..., v_k\}$ in V that are linearly independent and span V. If V has an ordered basis, then we define the **dimension of** V to be the number of elements in that basis.

Here are some elementary facts about bases that we will assume without proof.

- Every vector space has at least one basis.
- Every basis for a given vector space has the same size. (In particular, this is why we can define the dimension of a vector space to be the number of elements in any basis.)
- If the dimension of a vector space V is n, then any n linearly independent vectors are a basis and any n vectors that span V are a basis.
- If U is a subspace of V, then $\dim(U) \leq \dim(V)$.
- If U has a basis, that basis can be extended to a basis for V.

If a vector space has a finite basis, then it is finite-dimensional. But there are also infinite dimensional vector spaces as well. A vector space V is **infinite dimensional** if it cannot be spanned by finitely many vectors.

- The vector space $C(\mathbb{R})$ is infinite dimensional (Problem Set 5).
- The vector space of infinite sequences is infinite dimensional (Problem Set 6).

Problem 1

Show that $(1, 2, -1)$, $(2, -1, 1)$, and $(0, 1, 1)$ is a basis for $\mathbb{R}^3$. This shows that a vector space can have many different bases, but they all must have the same number of vectors.

Problem 2

Show that the set of polynomials in P_2 whose coefficients add to zero, which we will denote by Z_2, is a subspace of P_2. Find a basis for Z_2, and extend this to a basis for P_2.

Problem 3

Let S be the set of 2 by 2 symmetric matrices. Note that S is a subspace of $M_{2\times 2}$. Find a basis for S and extend this to a basis for $M_{2\times 2}$.

2 Rank, Nullity, and Rank-Nullity Theorem

Recall that for a linear transformation $T : V \to W$ between two vector spaces, the kernel of T is the set of vectors in v for which $T(v) = 0$ and the range is the set of possible values that can be attained in W. **It is an important fact that for any linear transformation, ker(T) is a subspace of V and range(T) is a subspace of W.**

Since $\ker(T)$ is a subspace of V and range(T) is a subspace of W, we can define their dimensions. We define

$$\text{nullity}(T) = \dim(\ker(T))$$

$$\text{rank}(T) = \dim(\text{range}(T))$$

It is an important fact that for every linear transformation $T : V \to W$,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

This important fact is called the **Rank-Nullity Theorem**.

Problem 4

Let V be any vector space and let $\dim(V) = n$. Define the identity transformation $I : V \to V$ by $I(v) = v$ for every v in V, and the zero transformation $Z : V \to V$ by $Z(v) = 0$ for every v in V. Find the rank and nullity of V and Z, and check that the Rank-Nullity Theorem holds.

Problem 5

Consider the derivative map $\frac{d}{dx} : P_n \to P_{n-1}$. What is the rank and nullity of $\frac{d}{dx}$? Check that the Rank-Nullity Theorem holds.

Problem 6

Consider the evaluation at $x = 0$ map $E_0 : P_n \to \mathbb{R}$. What is the rank and nullity of E_0? Check that the Rank-Nullity Theorem holds.

Problem 7

Consider the matrix transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(x_1, x_2, x_3) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

Find a basis for $\ker(T)$ and $\text{range}(T)$. Use this to check that the Rank-Nullity Theorem holds.

In particular, the previous problem considers the very important case of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by an m by n matrix A, defined by

$$T(\mathbf{x}) = A\mathbf{x}$$

As we saw from the previous problem, the **nullspace of** A ($\ker(T)$ where T is the associated transformation $\mathbf{x} \to A\mathbf{x}$) is given by the solution to the homogeneous system of equations $A\mathbf{x} = 0$, and **we can find a simple basis by writing the solution to this system in parametric vector form**. The range is given by the span of the columns of A, which is called the **column space of** A. How can we find a basis for the column space?

Note that **the span of the rows of a matrix is unchanged by row reduction**. So to find the column space, take the transpose of A, row reduce to reduced row echelon form, and then the nonzero rows of the matrix form a basis for the column space of the original matrix A. **You can check that the fact that a pivot column has zeros other than the pivot (which is a** 1**) in reduced row echelon form means that the nonzero rows in reduced row echelon form are linearly independent.**