Math 54: Linear Algebra and Differential Equations

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1 Subspaces

Recall that if V is a vector space and U is a subset of V, then U is a **subspace of** V if it is a vector space too, with the operations of vector addition and scalar multiplication taken from the larger space V.

Recall that to check if U is a subspace, it suffices to check the simpler condition that for all reals c_1, c_2, and for all vectors u_1, u_2 in U, we have that $c_1u_1 + c_2u_2$ is in U.

Examples of Subspaces:

- Recall that $M_{3\times3}$, the set of 3 by 3 matrices is a vector space. The subset $\text{Sym}_{3\times3}$ of symmetric 3 by 3 matrices is a subspace of $M_{3\times3}$, since for any two symmetric matrices A and B, $c_1A + c_2B$ is still a symmetric matrix.

- Recall that $\mathbb{R}^3$ is a vector space. Then, the set S of vectors $(x_1, x_2, 0)$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, since
$$c_1(a_1, a_2, 0) + c_2(b_1, b_2, 0) = (c_1a_1 + c_2b_1, c_1a_2 + c_2b_2, 0)$$
is still in S (since the last coordinate is still zero).

- Recall that P_n is a vector space. The set of polynomials with constant term equal to zero is a subspace, since $c_1p(x) + c_2q(x)$ has constant term equal to zero if $p(x)$ and $q(x)$ have constant term equal to zero.

- Every vector space is a subspace of itself.

- Let $\{0\}$ be the set of just the zero vector. Then $\{0\}$ is a subspace of any vector space. It is called the **trivial (zero) subspace**.

- The set of vectors in $\mathbb{R}^3$ with first coordinate equal to 1 is not a subspace of $\mathbb{R}^3$, since $2(1, 1, 1) = 2(1, 1, 1) + 0(1, 1, 1) = (2, 2, 2)$ which is a vector that does not have first coordinate equal to 1.

2 Span and Linear Independence

Recall the concepts of span and linear independence for vectors in $\mathbb{R}^n$. What we will see is that even though we defined these concepts for $\mathbb{R}^n$, we can actually extend the concepts of span and linear independence very naturally to arbitrary vector spaces.

First, let us define the concept of span for an arbitrary vector space.

Definition: Let $v_1, v_2, ..., v_k$ be vectors in an abstract vector space V. The **span** of $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}$ is the set of vectors in V of the form

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_k\mathbf{v_k}$$

where c_i are real numbers. We say that $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}$ **span** V if their span is the entire vector space V.

Let's consider some examples of span.

Problem 1

- Find the span of the polynomials 1, x, x^2 in P_4.
- Find the span of the polynomials 1, $1 + x$, and x^2 in P_4.
- Find the span of the matrices

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

  in $M_{2\times 2}$.
- Find a set of vectors in P_5 that spans P_5.

Now, let's analogously extend the idea of linear independence to an abstract vector space.

Definition: Let V be an arbitrary vector space. A set of vectors $v_1, v_2, ..., v_k$ is **linearly independent** if the only solution to

$$c_1v_1 + c_2v_2 + ... + c_kv_k = 0$$

is the trivial (zero) solution, $c_1, c_2, ..., c_k = 0$.

As a remark, note that while for vectors in $\mathbb{R}^n$, this was a linear system of equations, this is not necessarily the case for a general arbitrary vector space (at least not yet). However, many questions of linear independence in arbitrary vector spaces can be interpreted as linear systems.

Problem 2

Show that 1, x, and x^2 are linearly independent in P_3 in two different ways.

Problem 3

Show that $1 + 2 + x^2$, $3 - 4x$, and $1 + 2x$ are independent in P_4.

Problem 4

Show that the functions $y = 2^x$, $y = x^2$, and $y = x$ are linearly independent in $C(\mathbb{R})$.

3 Basis and Dimension

Let's think about $\mathbb{R}^n$ again. An easy set of vectors to think about in $\mathbb{R}^3$ for example is the set of vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Note that this set of vectors is **both** linearly independent and spans $\mathbb{R}^3$. So we say that this set is a **basis** for $\mathbb{R}^3$.

What is so useful about this particular basis for $\mathbb{R}^3$, known as the **standard basis**? Note that since these vectors span $\mathbb{R}^3$, every vector in $\mathbb{R}^3$ can be written as a linear combination of $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ and linear independence tells us that this linear combination is unique. For example,

$$(2, -3, 4) = 2(1, 0, 0) + (-3)(0, 1, 0) + 4(0, 0, 1)$$

and this is the unique way of expressing $(2, -3, 4)$ as a linear combination of the basis vectors. The fact that there are three linearly independent vectors in $\mathbb{R}^3$ that span $\mathbb{R}^3$ tells us that in some sense, $\mathbb{R}^3$ only has three distinct directions, so it makes sense to say that the dimension of $\mathbb{R}^3$ is 3, the number of elements in the basis.

We now extend this definition to arbitrary vector spaces in the following way.

Definition: Let V be an arbitrary vector space. An **ordered basis** $\mathcal{B}$ is an ordered list of vectors $\{v_1, v_2, ..., v_k\}$ in V that are linearly independent and span V. If V has an ordered basis, then we define the **dimension of** V to be the number of elements in that basis.

Here are some elementary facts about bases that we will assume without proof.

- Every vector space has at least one basis.
- Every basis for a given vector space has the same size. (In particular, this is why we can define the dimension of a vector space to be the number of elements in any basis.)
- If the dimension of a vector space V is n, then any n linearly independent vectors are a basis and any n vectors that span V are a basis.
- If U is a subspace of V, then $\dim(U) \leq \dim(V)$.
- If U has a basis, that basis can be extended to a basis for V.

If a vector space has a finite basis, then it is finite-dimensional. But there are also infinite dimensional vector spaces as well. A vector space V is **infinite dimensional** if it cannot be spanned by finitely many vectors.

- The vector space $C(\mathbb{R})$ is infinite dimensional (Problem Set 5).
- The vector space of infinite sequences is infinite dimensional (Problem Set 6).

Problem 5

Show that $(1, 2, -1)$, $(2, -1, 1)$, and $(0, 1, 1)$ is a basis for $\mathbb{R}^3$. This shows that a vector space can have many different bases, but they all must have the same number of vectors.

Problem 6

Show that the set of polynomials in P_2 whose coefficients add to zero, which we will denote by Z_2, is a subspace of P_2. Find a basis for Z_2, and extend this to a basis for P_2.

Problem 7

Let S be the set of 2 by 2 symmetric matrices. Note that S is a subspace of $M_{2\times 2}$. Find a basis for S and extend this to a basis for $M_{2\times 2}$.