Math 54: Linear Algebra and Differential Equations

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Recall from last time the invertible matrix theorem. **Invertible Matrix Theorem:** These following two statements are equivalent for a **square** matrix A.

- The square matrix A has an inverse A^{-1}.
- The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
- The homogeneous system $A\mathbf{x} = 0$ only has the trivial solution

Warning: The equivalence between statement 2 and 3 is only true for square matrices (Why?).

Remember from the equivalence theorem for linear independence that the third statement is equivalent to A having no free columns in its reduced row echelon form. But for a square matrix, this is equivalent to A having a pivot in each row, so that A **is invertible if and only if its reduced row echelon form is the identity matrix**.

How do you find the matrix inverse of a matrix A? Here are the steps.

- Make an augmented matrix with A on one side and the identity matrix of the same size on the other side $[A \mid I]$.
- Row reduce to make the side with A into reduced row echelon form.
- After this row reduction, if the left side of the line does not have a pivot in each row (is the identity matrix), the original matrix A does not have an inverse.
- If the left side of the line does have a pivot in each row (is the identity matrix), the inverse A^{-1} is what is on the right side of the line.

Problem 1

Use the algorithm above to show that the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$$

does not have a matrix inverse. Then, verify the invertible matrix theorem by showing that the homogeneous system

$$x_1 + x_2 + x_3 = 0$$

$$2x_1 - x_2 = 0$$

$$x_1 - 2x_2 - x_3 = 0$$

does not have a unique solution.

Problem 2

Assume that A and B are invertible. Is AB invertible? Is A^2 invertible? Is $(AB)^2$ is invertible?

Problem 3

Show that a matrix with two identical rows or two identical columns is not invertible.

Problem 4

Characterize all upper triangular matrices (only nonzero entries are in (i, j) positions where $j \geq i$) that are invertible.

Determinants

Remember that we are focusing on square matrices for now. Calculating the inverse of a matrix is tedious. That's why there is a function on matrices called the **determinant** that gives back a real number. The determinant of a matrix tells you instantly many properties of a matrix.

Calculating determinants is fairly simple. First, the determinant of any 1 by 1 matrix is just itself. So for example,

$$\begin{vmatrix} -1 \end{vmatrix} = -1$$

where the lines around the matrix mean determinant in this context, not absolute value.

Then, to calculate the determinant of an arbitrary matrix, use the following steps.

- Make an alternating sign chart the size of the matrix, starting with + in the $(1, 1)$ entry, and alternating to make a chart of the size of the matrix.
- Pick any row or column, ideally the easiest one that has the most zeros.
- For each entry in that row or column, multiply the entry by the sign in the sign chart, cross out the row and column of that entry to get a **cofactor** and multiply by the determinant of the cofactor.
- Do this for every entry in the row or column and add the results together.

This shows you for example that the determinant of any 2 by 2 matrix is given by the formula

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example:

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix}$$
$$= 1(2 \cdot 0 - 1 \cdot 1) - 2(0 \cdot 0 - 1 \cdot 1) + (-1)(0 \cdot 1 - 2 \cdot 1) = -1 + 2 + 2 = 3$$

Determinants can also be calculated by using elementary row operations, though each of the elementary row operations has an effect on the determinant.

- Switching rows changes the sign of the determinant.
- Adding a multiple of another row to a given row does NOT change the determinant.
- Multiply a row by a nonzero constant c multiplies the determinant by c.

If you want to simplify a matrix, I would suggest using the row operation of adding a multiple of one row to another, to get the matrix to have an easy row along which you can expand (for example, a row of mostly zeros).

Here is the reason that the determinant is important.
Theorem: The following two statements are equivalent for a square $n \times n$ matrix A.

- The determinant of A is nonzero.
- A is invertible.
- The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
- The homogeneous system $A\mathbf{x} = 0$ only has the trivial solution
- The row reduced echelon form of A is the identity matrix.
- The columns of A are linearly independent.
- The columns of A span $\mathbb{R}^n$.

Note that for these equivalences to hold, we must have that A is a square matrix. Which of these equivalences hold for general (non-square matrices A)?

Problem 5

Show that a matrix that has two identical rows has determinant equal to zero.

Problem 6

Calculate the determinant of an upper triangular matrix.