Math 54: Lecture 8/9/19

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Review of Complex Numbers

Remember that a complex number is a number of the form $a + bi$, where a and b are real numbers, and i denotes $\sqrt{-1}$. For a complex number $z = a + bi$, we denote the **complex conjugate** $\overline{z} = \overline{a + bi} = a - bi$, where the conjugate just negates the imaginary part. In addition, we denote the **norm** of a complex number $z = a + bi$ by

$$|z| = |a + bi| = (a^2 + b^2)^{1/2}$$

It is a fact that $|z|^2 = z\overline{z}$. Finally, complex conjugates behave nicely with respect to multiplication of complex numbers. In particular,

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$

which is an important fact that we will need.

We recall Euler's identity, which is a fundamental identity that states that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

By applying Euler's identity, we also have that

$$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$$
$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$
$$e^{-in\theta} = e^{i(-n\theta)} = \cos(-n\theta) + i\sin(-n\theta) = \cos(n\theta) - i\sin(n\theta)$$

where in the above calculations, we used the fact that cosine is even and sine is odd (so that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$).

Note that since

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

we have that the complex conjugate of $e^{i\theta}$ is $e^{-i\theta}$, so that

$$\overline{e^{i\theta}} = e^{-i\theta}$$

So when we take the complex conjugate of a complex exponential, we just negate the imaginary part of the exponent.

Finally, note that the function $e^{in\theta}$ is 2π-periodic. This is because

$$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

and both $\cos(n\theta)$ and $\sin(n\theta)$ are 2π-periodic.

1 Review of Inner Product Spaces

Let us recall two inner product spaces that we previously learned about. The first is $L^2([-\pi,\pi))$, which consists of all square integrable functions on $[-\pi,\pi)$ (where a square integrable function f on $[-\pi,\pi)$ is a function for which $\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta < \infty$). Because we will have to consider functions such as $e^{in\theta}$, which are complex valued (so when we put in a real number for θ, we get a complex number out), we will need to allow complex valued functions more generally to be in $L^2([-\pi,\pi))$.

Since we are considering complex valued functions more generally, we use the following inner product on $L^2([-\pi,\pi))$,

$$\langle f,g\rangle_{L^2([-\pi,\pi))} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}d\theta$$

So the norm squared here is

$$||f||^2_{L^2([-\pi,\pi))} = \langle f,f\rangle_{L^2([-\pi,\pi))} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^2 d\theta$$

where we used $|z|^2 = z\overline{z}$, so that $|f(\theta)|^2 = f(\theta)\overline{f(\theta)}$.

A function in $L^2([-\pi,\pi))$ can be considered as a function that is only defined on $[-\pi,\pi)$ and nowhere else. However, it will be useful to consider such functions also as 2π-periodic functions, where we take the function defined on $[-\pi,\pi)$ and extend it to be defined on all real numbers, by extending it to be periodic with period 2π. We will have to be able to think about these two different interpretations of functions in $L^2([-\pi,\pi))$ as both functions on $[-\pi,\pi)$ and as periodic functions on $\mathbb{R}$ with period 2π.

Evaluating the integrals for complex-valued functions over intervals is similar to just regular integrals, where we simply treat all i as constants.
Example: Calculate $\langle f,g\rangle_{L^2([-\pi,\pi))}$ where $f(x)=x$ and $g(x)=ix$.

To do this, we calculate

$$\begin{aligned}\langle f,g\rangle_{L^2([-\pi,\pi))} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\overline{ix}dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\overline{i}\overline{x}dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} -ix^2 dx = \frac{-i}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{-i}{2\pi}\cdot\frac{2\pi^3}{3} = -\frac{i\pi^2}{3}\end{aligned}$$

where we used that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, $\overline{i} = -i$, and since x takes on real values between $-\pi$ and π, $\overline{x} = x$.

Next, let us recall the inner product space $\ell^2(\mathbb{Z})$, which consists of all two sided sequences $..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...$ such that $\sum_{n\in\mathbb{Z}}|a_n|^2 = \sum_{n=-\infty}^{\infty}|a_n|^2$ is finite (where this sum is over all integers, not just positive ones). The inner product here was

$$\langle \mathbf{a},\mathbf{b}\rangle_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} a_n\overline{b_n}$$

so that the norm squared here is

$$||\mathbf{a}||^2_{\ell^2(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |a_n|^2$$

As a note, more generally, we can define $L^2([a,b))$ in an analogous way, with the general inner product

$$\langle f, g\rangle_{L^2([a,b))} = \frac{1}{b-a}\int_a^b f(\theta)\overline{g(\theta)}d\theta$$

where we can consider the functions here to be functions with period $b-a$. This is useful if we are considering functions that do not have a period of 2π, but have some other period. However, for the discussion below, we will consider just functions with period 2π.

Let's briefly recall some facts about inner product spaces. For a finite dimensional inner product space V, recall that it is nice if we can find an orthonormal basis $v_1, v_2, ..., v_n$ for V, which means that $\langle v_i, v_i\rangle =$ and $\langle v_i, v_j\rangle = 0$ when $i \neq j$. Because this is a basis, every vector can be written uniquely as a linear combination of the basis vectors, so that

$$v = c_1v_1 + c_2v_2 + ... + c_nv_n$$

uniquely. In addition, because the basis is orthonormal, we can calculate the coefficients c_i easily by simply taking an inner product:

$$c_i = \langle v, v_i\rangle$$

We will want to extend these ideas to the inner product space $L^2([-\pi,\pi))$ which will give us Fourier series. However, this inner product space, as we might suspect, is infinite dimensional, so any basis would have to be "infinite" in some sense. If we can find an "infinite" orthonormal basis $f_1, f_2, f_3, ...$ for this space, we could write any vector f in the space uniquely as an infinite linear combination of the basis elements,

$$f = c_1f_1 + c_2f_2 + c_3f_3 + ... = \sum_{i=1}^{\infty} c_if_i$$

where by orthonormality, we can get the coefficients c_i easily as

$$c_i = \langle f, f_i\rangle$$

(As a note, all of this is quite imprecise, but since this is not a proof-based class, we will not emphasize the details too much. In particular, one would need to rigorously define what a basis is for an infinite dimensional space, and what it means for the infinite sum above to converge.) Let us carry this out for $L^2([-\pi,\pi))$ now to get Fourier series.

Fourier Series

The key insight to Fourier series is that the collection of functions

$$f_n = e^{in\theta}$$

where n is any integer (not just positive, but all integers, including zero and negative integers) forms an orthonormal basis in $L^2([-\pi, \pi))$. For example,

$$f_1 = e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$f_0 = e^{i0\theta} = 1$$

$$f_{-1} = e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

Notice that all of the f_n are 2π-periodic.

Let us check that the f_n form an orthonormal set in $L^2([-\pi, \pi))$. We have that

$$\langle f_n, f_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\overline{e^{in\theta}}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 d\theta = 1$$

and if $m \neq n$,

$$\begin{aligned}\langle f_m, f_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta}\overline{e^{in\theta}}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{im\theta}e^{-in\theta}d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)\theta}d\theta \\ &= \frac{1}{2\pi i(m-n)} e^{i(m-n)\theta}\Big|_{-\pi}^{\pi} = 0\end{aligned}$$

where in the last step, we note that

$$\begin{aligned}e^{i(m-n)\pi} - e^{i(m-n)(-\pi)} &= (\cos((m-n)\pi) + i\sin((m-n)\pi)) - ((\cos((m-n)\pi) - i\sin((m-n)\pi)) \\ &= 2i\sin((m-n)\pi) = 0\end{aligned}$$

since sine of any multiply of π is zero. So in fact, $\{f_n\}_{n\in\mathbb{Z}}$ is an orthonormal set. We will not prove this, but it is also in fact an orthonormal basis in $L^2([-\pi, \pi))$.

Since f_n for integers n form an orthonormal basis in $L^2([-\pi, \pi))$, we can (formally) write any function f in $L^2([-\pi, \pi))$ as

$$f(\theta) \sim \sum_{n\in\mathbb{Z}} c_n f_n = \sum_{n\in\mathbb{Z}} c_n e^{in\theta} = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

where

$$c_n = \langle f, f_n \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{e^{in\theta}}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$$

We will denote the coefficients c_n by $\hat{f}(n)$, and we will call these the **Fourier coefficients**. So we have the **Fourier expansion**

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} \qquad \text{where } \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$$

(We put a $\sim$ instead of $=$ because it is not clear that the series on the other side actually converges, and if it does converge, it is not clear if it converges to $f(\theta)$ necessarily).

Let us note an important fact, called **Plancherel's theorem**. To motive this, note that given a function $f \in L^2([-\pi,\pi))$, we get the Fourier coefficients $\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta$, which in a sense, tell us how much of each frequency there is in the periodic function. We can list the Fourier series coefficients in an infinite two-sided sequence as follows:

$$..., \widehat{f}(-2), \widehat{f}(-1), \widehat{f}(0), \widehat{f}(1), \widehat{f}(2), ...$$

It is a fact that this is an element of $\ell^2(\mathbb{Z})$. It is a surprising result, called **Plancherel's theorem** that the Fourier series map that sends $f \in L^2([-\pi,\pi))$ to $..., \widehat{f}(-2), \widehat{f}(-1), \widehat{f}(0), \widehat{f}(1), \widehat{f}(2), ... \in \ell^2(\mathbb{Z})$ is an isometry. So it is bijective, and it preserves inner products, and hence norms. In particular, Plancherel's theorem states that

$$||f||^2_{L^2([-\pi,\pi))} = ||\widehat{f}||^2_{\ell^2(\mathbb{Z})}$$

so that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

This amazing fact can be used to prove many fundamental identities in math.

Let us end with a standard example found in many standard mathematical texts (see for example, Elias M. Stein and Rami Shakarchi's *Fourier Analysis: An Introduction (Princeton Lectures in Analysis I)*).

Example: Find the Fourier series expansion for $f = \theta$ defined on $[-\pi,\pi)$. Use this expansion to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

To do this, we simply calculate the Fourier series. It is advisable to calculate $\widehat{f}(0)$ first, and then calculate $\widehat{f}(n)$ for $n \neq 0$.

$$\widehat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-i0\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta d\theta = \frac{1}{2\pi}(0) = 0$$

and for $n \neq 0$, we can integrate by parts to get

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}d\theta = \frac{1}{2\pi}\left[-\frac{\theta e^{-in\theta}}{in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{e^{-in\theta}}{-in}d\theta\right] = \frac{1}{2\pi}\left[-\frac{\theta e^{-in\theta}}{in}\Big|_{-\pi}^{\pi} + \left(-\frac{e^{-in\theta}}{(in)^2}\right)\Big|_{-\pi}^{\pi}\right]$$

Let us calculate each of these terms.

$$-\frac{\theta e^{-in\theta}}{in}\Big|_{-\pi}^{\pi} = -\frac{1}{in}(\pi e^{-in\pi} - (-\pi)e^{in\pi}) = -\frac{\pi}{in}(e^{-in\pi} + e^{in\pi})$$

$$= -\frac{\pi}{in}(\cos(n\pi) - i\sin(n\pi) + (\cos(n\pi) + i\sin(n\pi))) = -\frac{\pi}{in}(2\cos(n\pi)) = -\frac{2\pi}{in}(-1)^n = \frac{2\pi(-1)^{n+1}}{in}$$

$$\left(-\frac{e^{-in\theta}}{(in)^2}\right)\Big|_{-\pi}^{\pi} = \frac{1}{n^2}(e^{-\in\pi} - e^{in\pi}) = \frac{1}{n^2}(\cos(n\pi) - i\sin(n\pi) - (\cos(n\pi) + i\sin(n\pi))$$

$$= \frac{1}{n^2}(-2i\sin(n\pi)) = 0$$

since sine of any multiple of π is zero. So we have that for $n \neq 0$,

$$\widehat{f}(n) = \frac{1}{2\pi}\left[-\frac{\theta e^{-in\theta}}{in}\Big|_{-\pi}^{\pi} + \left(-\frac{e^{-in\theta}}{(in)^2}\right)\Big|_{-\pi}^{\pi}\right] = \frac{1}{2\pi}\left[\frac{2\pi(-1)^{n+1}}{in} + 0\right] = \frac{(-1)^{n+1}}{in}$$

So

$$\widehat{f}(0) = 0 \qquad \widehat{f}(n) = \frac{(-1)^{n+1}}{in} \text{ for } n \neq 0$$

and thus the Fourier series for $f(\theta) = \theta$ on $[-\pi, \pi)$ is

$$f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{in\theta} = \sum_{n\neq 0, n\in\mathbb{Z}} \frac{(-1)^{n+1}}{in}e^{in\theta}$$

where the final sum is over all nonzero integers (both positive and negative).

By Plancherel's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2$$

The left hand side is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 d\theta = \frac{1}{2\pi}\frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

and the right hand side is

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 = \sum_{n\neq 0, n\in\mathbb{Z}} \left|\frac{(-1)^{n+1}}{in}\right|^2 = \sum_{n\neq 0, n\in\mathbb{Z}} \left(\frac{|(-1)^{n+1}|}{|i|\cdot|n|}\right)^2 = \sum_{n\neq 0, n\in\mathbb{Z}} \frac{1}{n^2} = 2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

where in the last step, we used that in the sum $\sum_{n\neq 0, n\in\mathbb{Z}} \frac{1}{n^2}$ which is over both positive and negative values, the terms for n and $-n$ give identical contributions, since $\frac{1}{n^2} = \frac{1}{(-n)^2}$. So by Plancherel's theorem,

$$2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3}$$

and hence,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

as desired.

As a final note about this example, in other disciplines such as engineering, it is useful to write Fourier series instead as sums of sines and cosines. Mathematically, this approach is less appealing because sines and cosines form an orthogonal, but not orthonormal basis. However, one can easily write a Fourier series as sines and cosines from the mathematical notation by separating out positive and negative n in the sum, using Euler's formula, and joining together the n and $-n$ terms. As an example, if we wanted to write the above Fourier series in terms of sines and cosines, we just calculate

$$\sum_{n\neq 0, n\in\mathbb{Z}} \frac{(-1)^{n+1}}{in} e^{in\theta} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} e^{in\theta} + \sum_{n=-\infty}^{-1} \frac{(-1)^{n+1}}{in} e^{in\theta}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} e^{in\theta} + \sum_{n=1}^{\infty} \frac{(-1)^{-n+1}}{-in} e^{-in\theta} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} e^{in\theta} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} (-e^{-in\theta})$$

$$\text{(since } (-1)^{-n+1} = (-1)^{-2n}(-1)^{n+1} = 1 \cdot (-1)^{n+1} = (-1)^{n+1}\text{)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} (e^{in\theta} - e^{-in\theta}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} (\cos(n\theta) + i\sin(n\theta) - (\cos(n\theta) - i\sin(n\theta)))$$

$$\text{(Euler's formula)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} (2i\sin(n\theta)) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(n\theta) \sim f(\theta)$$

So we see that our original sawtooth wave (since we extended it to be periodic with period 2π) can be written in some sense as an infinite sum of sine waves! So in essence, Fourier series say that we can "decompose" periodic signals into just sums of sines and cosines, where the decomposition tells us how much of each frequency is in the original signal.