Math 54: Lecture 8/8/19

Jeffrey Kuan

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More Systems of Differential Equations

Recall that we are trying to solve the system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Last time, we talked about the case of A having distinct eigenvalues and the case of A having complex conjugate eigenvalues. Let us first review the case of complex eigenvalues.

Problem 1

Solve the following system of differential equations.

$$x_1'(t) = -4x_1(t) + 5x_2(t)$$

$$x_2'(t) = -2x_1(t) + 2x_2(t)$$

The eigenvalues in the problem above are $\lambda = -1 \pm i$. Since the real part of these eigenvalues is negative, the equilibrium point at $(0, 0)$ is a stable equilibrium, since the associated exponential gives $e^{-(1\pm i)t}\mathbf{v}$ where $\mathbf{v}$ is an eigenvector. So this is $e^{-t}e^{it}\mathbf{v}$, and the e^{-t} makes the trajectory go towards the origin as $t \to \infty$.

Finally, there is a final case to consider. Let's consider the case where there is a repeated eigenvalue with algebraic multiplicity two, but with geometric multiplicity one. The problem in this case is that if λ is repeated twice as an eigenvalue, and there is only one dimension of eigenvectors, spanned by $\mathbf{v}$, then we cannot form two fundamental solutions as before.

We can still form one solution, as $\mathbf{v}e^{\lambda t}$. The problem is forming another solution that is linearly independent. As before, we can try multiplying by t, so we might try the vector-valued function $t\mathbf{v}e^{\lambda t}$. However, if we plug this into $\mathbf{x}' = A\mathbf{x}$, we will see that this is actually not a solution. In particular,

$$(t\mathbf{v}e^{\lambda t})' = \lambda \mathbf{v} t e^{\lambda t} + \mathbf{v}e^{\lambda t}$$

and

$$A(t\mathbf{v}e^{\lambda t}) = te^{\lambda t}A\mathbf{v} = \lambda \mathbf{v} t e^{\lambda t}$$

by the eigenvector condition. So we see that multiplying t to get $t\mathbf{v}e^{\lambda t}$ is not a solution. So there is more going on.

Let us instead try to guess something of the form

$$t\mathbf{v}e^{\lambda t} + \eta e^{\lambda t}$$

where η is a vector that will be chosen later. Let us plug this into the system and see what we need to choose η to be for this to be a solution.

$$(t\mathbf{v}e^{\lambda t} + \eta e^{\lambda t})' = \lambda \mathbf{v} t e^{\lambda t} + (\mathbf{v} + \lambda\eta)e^{\lambda t}$$

$$A(t\mathbf{v}e^{\lambda t} + \eta e^{\lambda t}) = \lambda \mathbf{v} t e^{\lambda t} + A\eta e^{\lambda t}$$

So for this to be a solution, we need to choose η so that

$$A\eta = \mathbf{v} + \lambda\eta$$

So η must satisfy

$$(A - \lambda I)\eta = \mathbf{v}$$

and where $\mathbf{v}$ is the eigenvector for λ.

So for repeated eigenvalues with only one basis of eigenvectors, our fundamental solution set is

$$\mathbf{f_1}(t) = ve^{\lambda t}$$

$$\mathbf{f_2}(t) = t\mathbf{v}e^{\lambda t} + \eta e^{\lambda t}$$

where v is an eigenvector of λ and η is a vector satisfying

$$(A - \lambda I)\eta = \mathbf{v}$$

Let us solve the system

$$x_1'(t) = -x_1(t) + 2x_2(t)$$

$$x_2'(t) = -2x_1(t) + 3x_2(t)$$

We can rewrite this system as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

We can calculate that $A = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ has $\lambda = 1$ as a repeated eigenvalue.

$$A - I = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$$

So we can take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and there is only one dimension of eigenvectors.

So we have that one fundamental solution is

$$\mathbf{f_1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$$

We next need to find η such that

$$(A - \lambda I)\eta = \mathbf{v}$$

where $\lambda = 1$. So we need to solve $(A - I)\eta = \mathbf{v}$, which is

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \eta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We just need to choose any solution η to this. We can take for example $\eta = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$. So we can take the second fundamental solution to be

$$\mathbf{f_2}(t) = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} e^t = \begin{bmatrix} te^t \\ \left(t + \frac{1}{2}\right) e^t \end{bmatrix}$$

So our solution here is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} te^t \\ \left(t + \frac{1}{2}\right) e^t \end{bmatrix}$$

So we have

$$x_1(t) = c_1 e^t + c_2 t e^t$$
$$x_2(t) = c_1 e^t + c_2 \left(t + \frac{1}{2}\right) e^t$$

Problem 2

Solve the system

$$x_1'(t) = x_1(t) - 3x_2(t)$$
$$x_2'(t) = 3x_1(t) - 5x_2(t)$$

Higher Order Differential Equations

We can more generally solve any higher order differential equation in terms of systems. This is by a clever trick where we record all derivatives up to the $n-1$ derivative as data in a vector-valued variable $\mathbf{x}(t)$. Let us see this first in the context of a second order equation.

Let us consider the differential equation first for a simple pendulum, given by

$$\frac{d^2\theta}{dt^2} + \theta = 0$$

where we used the small angle approximation $\sin(\theta) \approx \theta$ (we set all physical constants to 1 for simplicity). Let's see how we can convert this into a system.

Define

$$\mathbf{x}(t) = \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix}$$

Note then that

$$(\theta(t))' = \theta'(t) \text{ by definition}$$

$$(\theta'(t))' = -\theta \text{ by the original differential equation}$$

so we can write our original second order system as a system of differential equations of the form

$$\frac{d}{dt} \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \theta'(t) \end{bmatrix}$$

We can then solve this using a system and graph the phase portrait. Note that the phase portrait shows the motion of the pendulum by showing at any moment its instantaneous position and velocity (so the trajectory on the phase portrait gives both the position and **momentum** of the pendulum). The phase portrait consists of ellipses.

Problem 3

Solve the simple pendulum system and graph the phase portrait. Interpret the phase portrait trajectories physically.

Another interesting phenomenon associated with oscillations is that of damped oscillations, where there is a dissipative nonconservative force, such as friction, that decreases the total energy of the system. A differential equation that might model a damped oscillation or a pendulum with damping might be something like

$$\frac{d^2\theta}{dt^2} = -\theta - \frac{d\theta}{dt}$$

where the farther from the equilibrium $\theta = 0$ and the faster the pendulum is moving (the larger θ' is), the more negative the angular acceleration is, which tends to bring the pendulum back to the equilibrium position $\theta = 0$.

Problem 4

Convert the damped oscillation second order equation above into a system to solve the equation. Graph the phase portrait. This should be a spiral going in towards the stable equilibrium at the origin.