Math 54: Lecture 8/7/19

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August 7, 2019

An Introduction to Systems of Differential Equations

Example: Suppose that we have two populations, one of gazelles and one of lions. Let the number of gazelles and the number of lions at time t be denoted by $x_1(t)$ and $x_2(t)$ respectively. Here are some heuristics of the lion and gazelle population.

- The more gazelles there are, the faster the population grows. Same for lions.
- The more lions there are, the faster the population of gazelles decreases, since the lions prey on the gazelles.
- The more gazelles there are, the faster the population of lions increases, since there are more resources for the lion population.

Using these heuristics, a reasonable model for such a **predator-prey system** might be something like

$$x_1'(t) = x_1(t) - x_2(t)$$
$$x_2'(t) = x_1(t) + x_2(t)$$

where the differential equations governing the gazelle and lion populations are **coupled** to each other. This is called a **first order system of differential equations**.

We can write such a system in matrix form as follows. Using matrix multiplication, we can express the above system as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Let us define the solution of this system to be the vector valued function

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where this is a column vector that changes with time, where the first row tells us the gazelle population at time t and the second row tells us the lion population at time t. We can differentiate such a vector-valued function of time by just differentiating each component separately. Therefore, we can write the system above as

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t)$$

More generally, a first order system of differential equations with the same number of equations as variables is a differential equation of the form

$$\mathbf{x}'(t) = A\mathbf{x}(t) \quad \text{ where } A \text{ is an } n \text{ by } n \text{ matrix}$$

Written out completely, this is

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

For any system of differential equations, if there are n variables, it is a fact that the solution space has dimension n. So to find a solution set, we need n linearly independent functions that are solutions. Given any n vector-valued functions $\mathbf{f_1}(t), \mathbf{f_2}(t), ..., \mathbf{f_n}(t)$ that are solutions of the system, we have that this set is a basis for the solution space if and only the functions are linearly independent for all t, or if and only if the matrix which has $\mathbf{f_j}(t)$ in the jth column has nonzero determinant for every time t. In this case, we call the functions $\mathbf{f_1}(t), \mathbf{f_2}(t), ..., \mathbf{f_n}(t)$ a **fundamental solution set** for the system of differential equations.

So our goal in solving a system of differential equations is to find a fundamental solution set. We will consider mostly the case of 2 by 2 systems, in which case we need to find two linearly independent vector-valued functions that are solutions.

Before talking about how to do this, let us talk about how to visualize the solutions to these systems of differential equations. We can do what is called a **phase plane analysis** to visualize solutions. This is similar to the method of slope fields, except there is no time axis (in slope fields, there is an axis for time, but for a phase plane, the two axes are the dependent variables $x_1(t)$ and $x_2(t)$ and the time is represented by parametrization along what are called the integral curves.)

For each point (a, b), we can calculate the slope $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$ of an integral curve that passes through (a, b). We can then draw an x_1, x_2 plane and then at the point (a, b), draw an arrow that represents the slope $\left.\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}\right|_{(a,b)} = A \begin{bmatrix} a \\ b \end{bmatrix}$. Then, we can draw the integral curves of the resulting vector field, which represent the solutions to the system.

Solving Systems of Differential Equations

Let us learn how to solve systems of first-order differential equations. We will see that the spectral theory of the matrix A will play an important role here. We will first start with homogeneous systems, which in physical applications typically represent equilibrium states of systems.

So let us consider a 2 by 2 system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where A is 2 by 2. Suppose that A has two distinct real eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$. Then, note that the function $\mathbf{v_1}e^{\lambda_1 t}$ is a solution to the system, since

$$\frac{d}{dt}(\mathbf{v_1}e^{\lambda_1 t}) = \lambda_1(\mathbf{v_1}e^{\lambda_1 t})$$

$$A(\mathbf{v_1}e^{\lambda_1 t}) = (A\mathbf{v_1})e^{\lambda_1 t} = \lambda_1(v_1 e^{\lambda_1 t})$$

Similarly, $\mathbf{v_2}e^{\lambda_2 t}$ is a solution to the system too, and one can check that these solutions are linearly independent for all t so that they form a fundamental solution set. So then, the general solution in this case is

$$\mathbf{x}(t) = c_1\mathbf{v_1}e^{\lambda_1 t} + c_2\mathbf{v_2}e^{\lambda_2 t}$$

Problem 1

Solve the following system of differential equations.

$$x_1'(t) = 3x_1(t) - x_2(t)$$

$$x_2'(t) = -x_1(t) + 3x_2(t)$$

$$\begin{aligned} x_1'(t) &= x_1(t) + 3x_2(t) \\ x_2'(t) &= x_1(t) - x_2(t) \end{aligned}$$

Next, let's consider the second situation, where we have two complex conjugate eigenvalues. It is a fact that when any non-real eigenvalues come out as complex conjugates, and their eigenvectors are complex conjugates of each other. Let us consider the example of the original predator prey system we considered at the start.

$$\begin{aligned} x_1'(t) &= x_1(t) - x_2(t) \\ x_2'(t) &= x_1(t) + x_2(t) \end{aligned}$$

We can express this system as

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t)$$

Motivated by the procedure before, let us try to find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We get that the characteristic polynomial is

$$\text{char}_A(x) = (1-x)^2 + 1 = x^2 - 2x + 2$$

So the eigenvalues here are $\lambda = 1 \pm i$. Let us calculate the eigenvectors for each eigenvalue.

For $\lambda = 1 + i$, we get

$$A - \lambda I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

so a basis for the eigenspace here is $\begin{bmatrix} i \\ 1 \end{bmatrix}$. For $\lambda = 1 - i$, we get

$$A - \lambda I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$$

so a basis for the eigenspace here is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. Note that the eigenvalues and the associated eigenvectors are complex conjugates of each other. This will always be the case.

When we get complex conjugate eigenvalues, the way to get a fundamental solution set is to take one of the conjugate eigenvalues, and consider $\mathbf{v_1}(t)e^{\lambda_1 t}$ as before. In this case, taking $\lambda = 1 + i$, we get

$$e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The trick to get a fundamental solution set mirrors our derivation for second order linear homogeneous differential equations. We want to use Euler's formula to expand the complex exponential, where we recall that Euler's formula is $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Then, separate out all the real terms into one vector and all of the complex terms into the other vector. In this case, we get

$$\begin{aligned} e^{(1+i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} = e^t e^{it} \begin{bmatrix} i \\ 1 \end{bmatrix} &= e^t(\cos(t) + i\sin(t)) \begin{bmatrix} i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^t\sin(t) + ie^t\cos(t) \\ e^t\cos(t) + ie^t\sin(t) \end{bmatrix} = \begin{bmatrix} -e^t\sin(t) \\ e^t\cos(t) \end{bmatrix} + i \begin{bmatrix} e^t\cos(t) \\ e^t\sin(t) \end{bmatrix} \end{aligned}$$

Then the "real" and "imaginary" part of the final result give a fundamental solution set for our original differential equation! In particular, the fundamental solution set would be

$$\mathbf{f_1(t)} = \begin{bmatrix} -e^t\sin(t) \\ e^t\cos(t) \end{bmatrix} \quad \mathbf{f_2(t)} = \begin{bmatrix} e^t\cos(t) \\ e^t\sin(t) \end{bmatrix}$$

So then the solution to the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -e^t\sin(t) \\ e^t\cos(t) \end{bmatrix} + c_2 \begin{bmatrix} e^t\cos(t) \\ e^t\sin(t) \end{bmatrix}$$

Remembering that $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, we have that

$$x_1(t) = -c_1e^t\sin(t) + c_2e^t\cos(t)$$

$$x_2(t) = c_1e^t\cos(t) + c_2e^t\sin(t)$$