Math 1B: Homework 2 Hints

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1 Hint for Question 37, Section 7.4

We use partial fraction decomposition:

$$\int \frac{x^2-3x+7}{(x^2-4x+6)^2}dx$$

The partial fraction decomposition of the integrand is

$$\frac{x^2-3x+7}{(x^2-4x+6)^2} = \frac{Ax+B}{x^2-4x+6} + \frac{Cx+D}{(x^2-4x+6)^2}$$

(How do you know x^2-4x+6 is an irreducible quadratic?)

Multiply both sides by $(x^2-4x+6)^2$. So

$$x^2-3x+7 = (Ax+B)(x^2-4x+6)+Cx+D$$

We can either plug in values of x, or expand the sides and match the coefficients. (Sometimes, this strategy of matching coefficients is easier)

$$x^2-3x+7 = Ax^3+(-4A+B)x^2+(6A-4B+C)x+6B+D$$

Matching coefficients,

$$A = 0$$
$$-4A+B = 1$$
$$6A-4B+C = -3$$
$$6B+D = 7$$

If we solve for A, B, C, D, we get $A = 0$, $B = 1$, $C = 1$, and $D = 1$. So

$$\int \frac{x^2-3x+7}{(x^2-4x+6)^2}dx = \int \frac{1}{x^2-4x+6}dx + \int \frac{x+1}{(x^2-4x+6)^2}dx$$

As a hint, strategically break this integral into three integrals.

$$\int \frac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2} dx = \int \frac{1}{x^2 - 4x + 6} dx + \int \frac{x + 1}{(x^2 - 4x + 6)^2} dx$$
$$= \int \frac{1}{x^2 - 4x + 6} dx + \int \frac{x - 2}{(x^2 - 4x + 6)^2} dx + \int \frac{3}{(x^2 - 4x + 6)^2} dx$$

Hint for the first integral:

$$\int \frac{1}{x^2 - 4x + 6} dx$$

Complete the square:

$$\int \frac{1}{x^2 - 4x + 6} dx = \int \frac{1}{(x - 2)^2 + 2} dx = \frac{1}{2} \int \frac{1}{\left(\frac{x-2}{\sqrt{2}}\right)^2 + 1} dx$$

Set $u = \frac{x-2}{\sqrt{2}}$.

Hint for the second integral:

$$\int \frac{x - 2}{(x^2 - 4x + 6)^2} dx$$

Set $u = x^2 - 4x + 6$. Then, $du = (2x - 4)dx$, so $\frac{1}{2}du = (x - 2)dx$.

Hint for the last integral: Complete the square as follows.

$$\int \frac{3}{(x^2 - 4x + 6)^2} dx = \int \frac{3}{((x - 2)^2 + 2)^2} dx$$

Do a trigonometric substitution, where the hypotenuse of the right triangle is $\sqrt{(x - 2)^2 + 2}$ and the leg lengths are $x - 2$ and $\sqrt{2}$.

2 Hint for Question 72, Section 7.4

$$\int \frac{1}{(x^2 + a^2)^n} dx = \frac{1}{a^2} \int \frac{a^2}{(x^2 + a^2)^n} dx = \frac{1}{a^2} \int \left(\frac{x^2 + a^2}{(x^2 + a^2)^n} dx - \frac{x^2}{(x^2 + a^2)^n} \right) dx$$
$$= \frac{1}{a^2} \int \frac{1}{(x^2 + a^2)^{n-1}} dx - \frac{1}{a^2} \int \frac{x^2}{(x^2 + a^2)^n} dx$$

For the second integral,

$$\int \frac{x^2}{(x^2 + a^2)^n} dx$$

integrate by parts with

$$u = x$$
$$dv = \frac{x}{(x^2 + a^2)^n} dx$$

Note that we can integrate dv by using the substitution $w = x^2 + a^2$.

3 Hint for Question 48, Section 7.2

Multiply top and bottom by $\cos(x) + 1$. Then,

$$\int \frac{1}{\cos(x) - 1} dx = \int \frac{\cos(x) + 1}{(\cos(x) - 1)(\cos(x) + 1)} dx = \int \frac{\cos(x) + 1}{\cos^2(x) - 1} dx$$
$$= \int \frac{\cos(x) + 1}{-\sin^2(x)} dx = -\int \frac{\cos(x)}{\sin^2(x)} dx - \int \csc^2(x) dx$$

The first integral is doable using a u-substitution. For the second integral, use $\csc^2(x) = 1 + \cot^2(x)$.

4 Hint for Question 43, Section 7.3

Let the circle of radius R be described by $x^2 + y^2 = R^2$. The center of the circle of radius r is a length $\sqrt{R^2 - r^2}$ above the center of the circle of radius R (Why?). So the equation of the smaller circle is

$$x^2 + (y - \sqrt{R^2 - r^2})^2 = r^2$$

The two equations can be rewritten as $x^2 = R^2 - y^2$ and $x^2 = r^2 - (y - \sqrt{R^2 - r^2})^2$. The y-value at which the two circles intersect is given by solving the equation

$$R^2 - y^2 = r^2 - (y - \sqrt{R^2 - r^2})^2$$

Then, once you solve for the y value, you can find the x value by back-substituting into $x^2 + y^2 = R^2$. Then, it is easy to set up the integral. Solve the integral by trigonometric substitution.