Math 54 Final Exam (Practice 4)

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Name: Answer Key

SSID:

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- *Good luck!*

Problem 1 (30 points)

Recall that P_4 is the set of polynomials with real coefficients of degree ≤ 4.

Part (a)

Let U be the set of polynomials in P_4 such that $p(0) = 0$ and $p(1) = -1$. Is U a subspace of P_4? If so, prove that U is a subspace, and find a basis for U. If not, give a proof that shows that U is not a subspace. [15 points]

No. U does not contain the zero polynomial (which is the zero vector) and every subspace must contain the zero vector.

Part (b)

Let V be the set of polynomials in P_4 such that $p(-2) = 0$ and $p(1) = 0$. Is U a subspace of P_4? If so, prove that V is a subspace, and find a basis for V. If not, give a proof that shows that V is not a subspace. [15 points]

Yes. If $p_1, p_2 \in V$, $c_1p_1 + c_2p_2 \in V$ since

$$(c_1p_1+c_2p_2)(-2) = c_1p_1(-2) + c_2p_2(-2) = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

$$(c_1p_1+c_2p_2)(1) = c_1p_1(1) + c_2p_2(1) = c_1 \cdot 0 + c_2 \cdot 0 = 0. \checkmark$$

$$a + bx + cx^2 + dx^3 + ex^4$$

$$a - 2b + 4c - 8d + 16e = 0$$
$$a + b + c + d + e = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 4 & -8 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & -3 & 9 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -2 & 6 \\ 0 & 1 & -1 & 3 & -5 \end{bmatrix}$$

$$x = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -6 \\ 5 \\ 0 \\ 0 \\ 9 \end{bmatrix}$$

$$\boxed{\text{Basis: } -2+x+x^2,\; 2-3x+x^3,\; -6+5x+x^4}$$

Problem 2 (30 points)

Consider the map $T : P_2 \to M_{2\times 2}$ given by

$$T(p(x)) = \begin{bmatrix} p(-1) & p(1) \\ p'(-1) & p'(1) \end{bmatrix}$$

Part (a)

Prove that T is a linear transformation. [5 points]

$$T(c_1p_1+c_2p_2) = \begin{bmatrix} (c_1p_1+c_2p_2)(-1) & (c_1p_1+c_2p_2)(1) \\ (c_1p_1+c_2p_2)'(-1) & (c_1p_1+c_2p_2)'(1) \end{bmatrix}$$

$$= \begin{bmatrix} c_1p_1(-1)+c_2p_2(-1) & c_1p_1(1)+c_2p_2(1) \\ c_1p_1'(-1)+c_2p_2'(-1) & c_1p_1'(1)+c_2p_2'(1) \end{bmatrix}$$

$$= c_1\begin{bmatrix} p_1(-1) & p_1(1) \\ p_1'(-1) & p_1'(1) \end{bmatrix} + c_2\begin{bmatrix} p_2(-1) & p_2(1) \\ p_2'(-1) & p_2'(1) \end{bmatrix}$$

$$= c_1T(p_1)+c_2T(p_2) \checkmark$$

Part (b)

Find a basis for ker(T) and a basis for range(T). [25 points]

$$T(a+bx+cx^2) = \begin{bmatrix} a-b+c & a+b+c \\ b-2c & b+2c \end{bmatrix}$$

$(a+bx+cx^2)' = b+2cx$

$$a-b+c=0$$
$$a+b+c=0$$
$$\left.\begin{matrix} b-2c=0 \\ b+2c=0 \end{matrix}\right\} \to b=0 \quad c=0 \Rightarrow a=0$$

So $\ker(T)=\{0\}$
so basis for $\ker(T)$ is empty.

By Rank-Nullity, $\underbrace{\text{rank}(T)}_{=\dim(\text{range}(T))}=3$ (since nullity$(T)=0$, $\dim(P_2)=3$).

Note $T(1)=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $T(x)=\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $T(x^2)=\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$

are in range (T) and are linearly independent (as one can check).

So they are a basis for range(T).

$$\boxed{\text{Basis for range}(T): \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}}$$

Problem 3 (30 points)

Consider the matrix

$$A = \begin{bmatrix} x & x+2 & 0 \\ x+2 & 2x+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Part (a)

Find all x such that the matrix A is positive definite. [15 points]

Sylvester's Criterion

$x > 0$

$$\begin{vmatrix} x & x+2 \\ x+2 & 2x+1 \end{vmatrix} = 2x^2 + x - x^2 - 4x - 4$$

$$= x^2 - 3x - 4 = (x-4)(x+1) > 0$$

$$\Downarrow$$

$$x < -1 \text{ or } x > 4$$

$$\begin{vmatrix} x & x+2 & 0 \\ x+2 & 2x+1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (x-4)(x+1) > 0 \Rightarrow$$

$x > 0$ and $(x < -1 \text{ or } x > 4) \Rightarrow \boxed{x > 4}$

Part (b)

Is the matrix A positive semidefinite when $x = 0$? Justify your answer with a proof or a computation. [15 points]

$x = 0$ $\quad A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} -x & 2 & 0 \\ 2 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = (1-x)(x^2 - x - 4)$$

$\lambda = 1 \qquad \lambda = \frac{1 \pm \sqrt{17}}{2}$

$\lambda = \frac{1 - \sqrt{17}}{2} < 0.$

$\boxed{\text{So } A \text{ is not positive semidefinite when } x = 0.}$

Problem 4 (30 points)

Consider the subspace W of points $(x, y, z) \in \mathbb{R}^3$ such that $-3x + 2y - z = 0$.

Part (a)

Find an orthonormal basis for W. [10 points]

$-3x + 2y - z = 0$

Basis: $\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$w_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ $\quad v_2 - \langle v_2, w_1 \rangle w_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{10}(6) \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{5} \\ 1 \\ \frac{1}{5} \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{35}} \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

$\boxed{w_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \; w_2 = \frac{1}{\sqrt{35}} \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}}$

Part (b)

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(v) = \mathrm{proj}_{W^\perp}(v)$$

Find the matrix $[T]_{\mathcal{B} \to \mathcal{B}}$ of T with respect to the standard basis $\mathcal{B}$ for $\mathbb{R}^3$.

Note that $W^\perp$ is spanned by $(-3, 2, -1)$.
(Any vector $(x, y, z) \in W$ has

$-3x + 2y - z = 0 \Rightarrow (-3, 2, -1) \cdot \underbrace{(x, y, z)}_{\in W} = 0$

and $\dim(W^\perp) = 1$).

$w_3 = \frac{1}{\sqrt{14}} \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$ is an ONB for $W^\perp$.

$\mathrm{proj}_{W^\perp}(1,0,0) = \langle e_1, w_3 \rangle w_3 = \frac{1}{14}(-3) \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{9}{14} \\ -\frac{3}{7} \\ \frac{3}{14} \end{pmatrix}$

$\mathrm{proj}_{W^\perp}(0,1,0) = \langle e_2, w_3 \rangle w_3 = \frac{1}{14}(2) \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{7} \\ \frac{2}{7} \\ -\frac{1}{7} \end{pmatrix}$

$\mathrm{proj}_{W^\perp}(0,0,1) = \langle e_3, w_3 \rangle w_3$
$= \frac{1}{14}(-1) \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{14} \\ -\frac{1}{7} \\ \frac{1}{14} \end{pmatrix}$

$$\boxed{[T]_{\mathcal{B} \to \mathcal{B}} = \begin{bmatrix} \frac{9}{14} & -\frac{3}{7} & \frac{3}{14} \\ -\frac{3}{7} & \frac{2}{7} & -\frac{1}{7} \\ \frac{3}{14} & -\frac{1}{7} & \frac{1}{14} \end{bmatrix}}$$

Problem 5 (20 points)

Find the general solution to the following system.

$$x_1'(t) = x_1(t) + x_3(t)$$
$$x_2'(t) = x_2(t) + x_3(t)$$
$$x_3'(t) = x_1(t) + x_2(t) - x_3(t)$$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad \vec{x}\,' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \vec{x}$$

$$\begin{vmatrix} 1-x & 0 & 1 \\ 0 & 1-x & 1 \\ 1 & 1 & -1-x \end{vmatrix}$$

$$= (1-x)(-1+x^2-1) + 1(x-1)$$

$$= (x-1)(1+2-x^2) = (x-1)(3-x^2)$$

$$\lambda = 1,\ \lambda = \sqrt{3},\ \lambda = -\sqrt{3}$$

$$\lambda = 1 \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = v_1$$

$$\lambda = \sqrt{3} \quad \begin{bmatrix} 1-\sqrt{3} & 0 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -1-\sqrt{3} \end{bmatrix} \to \begin{bmatrix} 1-\sqrt{3} & 0 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 1 \\ -1+\sqrt{3} \end{pmatrix}$$

$(1-\sqrt{3})R_3 - R_1 - R_2$

$(-1-\sqrt{3})(1-\sqrt{3}) = 2$

$$\lambda = -\sqrt{3} \quad \begin{bmatrix} 1+\sqrt{3} & 0 & 1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -1+\sqrt{3} \end{bmatrix} \to \begin{bmatrix} 1+\sqrt{3} & 0 & 1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_3(1+\sqrt{3}) - R_1 - R_2$

$(-1+\sqrt{3})(1+\sqrt{3}) = 2$

$$V_3 = \begin{pmatrix} 1 \\ 1 \\ -1-\sqrt{3} \end{pmatrix}$$

$$\boxed{\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{\sqrt{3}t} \begin{pmatrix} 1 \\ 1 \\ -1+\sqrt{3} \end{pmatrix} + C_3 e^{-\sqrt{3}t} \begin{pmatrix} 1 \\ 1 \\ -1-\sqrt{3} \end{pmatrix}}$$

Problem 6 (40 points)

Find the Fourier series expansion for

$$f(\theta) = |\theta|, \quad -\pi < x < \pi$$

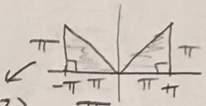
def of $|\theta|$ $\downarrow$ $-\theta, \ -\pi < x < 0$; $\theta, \ 0 < x < \pi$

Use Plancherel's theorem to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \cancel{1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} \cdots} \quad 1 + \frac{1}{81} + \frac{1}{625} + \ldots$$

(Hint: You should get $\widehat{f}(n) = \frac{\cos(n\pi)-1}{\pi n^2}$ for $n \neq 0$. Don't forget to also calculate $\widehat{f}(0)$.)

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$



$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|\, d\theta = \frac{1}{2\pi}(\pi^2) = \frac{\pi}{2}$$

$n \neq 0$ $\quad \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta| e^{-in\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{0} -\theta e^{-in\theta}\, d\theta + \frac{1}{2\pi}\int_{0}^{\pi} \theta e^{-in\theta}\, d\theta$

$$= \frac{1}{2\pi}\left(\frac{\theta e^{-in\theta}}{in}\Big|_{-\pi}^{0} + \frac{e^{-in\theta}}{(in)^2}\Big|_{-\pi}^{0} + \left(-\frac{\theta e^{-in\theta}}{in}\right)\Big|_{0}^{\pi} + \left(\frac{-e^{-in\theta}}{(in)^2}\right)\Big|_{0}^{\pi} \right)$$

ASIDE: $\int \theta e^{-in\theta} d\theta \quad u = \theta \quad dv = e^{-in\theta} d\theta \quad du = d\theta \quad v = \frac{e^{-in\theta}}{-in}$

$$= -\frac{\theta e^{-in\theta}}{in} + \int \frac{e^{-in\theta}}{in}\, d\theta$$

$$= -\frac{\theta e^{-in\theta}}{in} + \frac{e^{-in\theta}}{-(in)^2}$$

$$= \frac{1}{2\pi}\left(\underbrace{\cancel{\frac{\pi e^{in\pi}}{in}}}_{①} + \underbrace{\frac{1}{(in)^2} - \frac{e^{in\pi}}{(in)^2}}_{②} + \underbrace{\left(-\cancel{\frac{\pi e^{-in\pi}}{in}}\right)}_{①} - \underbrace{\frac{e^{-in\pi}}{(in)^2} + \frac{1}{(in)^2}}_{②} \right)$$

① $\frac{\pi e^{in\pi}}{in} - \frac{\pi e^{-in\pi}}{in} = \frac{\pi}{in}\left(\cancel{\cos(n\pi)} + \cancel{i\sin(n\pi)} - \cancel{\cos(n\pi)} + i\cancel{\sin(n\pi)}\right) = 0$

② $-\frac{e^{in\pi}}{(in)^2} - \frac{e^{-in\pi}}{(in)^2} = \frac{1}{n^2}\left(\cos(n\pi) + \cancel{i\sin(n\pi)} + \cos(n\pi) - \cancel{i\sin(n\pi)}\right)$

$$= \frac{2\cos(n\pi)}{n^2}$$

$\hat{f}(n)$ for $n \neq 0$ $= \frac{1}{2\pi}\left(\frac{2\cos(n\pi)}{n^2} - \frac{2}{n^2}\right) = \frac{\cos(n\pi)-1}{\pi n^2}$

$$\boxed{f(\theta) = \frac{\pi}{2} + \sum_{n\neq 0,\, n\in\mathbb{Z}} \left(\frac{\cos(n\pi)-1}{\pi n^2}\right) e^{in\theta}}$$

$$= \frac{(-1)^n - 1}{\pi n^2}$$

$$= \begin{cases} 0 & \text{if } n \text{ even} \\ -\frac{2}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

Plancherel's Theorem: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$

$\Downarrow$ since $\frac{4}{\pi^2 n^4} = \frac{4}{\pi^2(-n)^4}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 d\theta = \frac{\pi^2}{3}$$

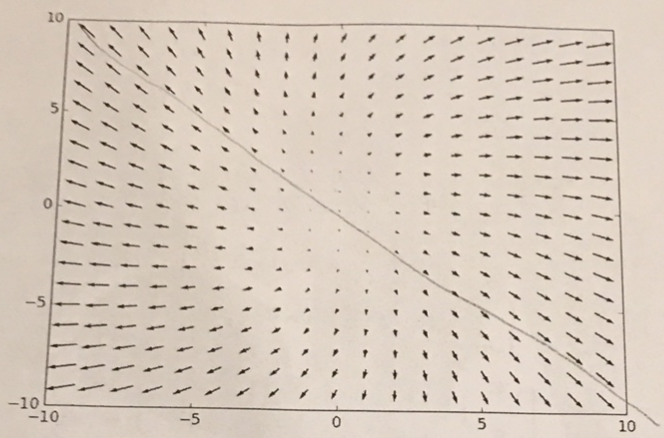
$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \underbrace{\frac{\pi^2}{4}}_{n=0} + \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} |\hat{f}(n)|^2$$

$$= \frac{\pi^2}{4} + \sum_{\substack{n\neq 0\\ n \text{ odd}}} \frac{4}{\pi^2 n^4} = \frac{\pi^2}{4} + 2\sum_{\substack{n \text{ odd}\\ n>0}} \frac{4}{\pi^2 n^4}$$

$$= \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{n \text{ odd},\, n>0} \frac{1}{n^4}$$

So $\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{n \text{ odd},\, n>0} \frac{1}{n^4}$

$$\Rightarrow \boxed{\frac{\pi^4}{96} = \sum_{n \text{ odd},\, n>0} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}}$$



No. This is not the correct phase portrait since the phase portrait depicts a system with only one eigenvector $\begin{pmatrix}1\\-1\end{pmatrix}$ in the basis for the eigenspace but the eigenspace for our system below is spanned by $\begin{pmatrix}1\\1\end{pmatrix}$, not $\begin{pmatrix}1\\-1\end{pmatrix}$.

Problem 7 (20 points)

Consider the system of differential equations given by

$$x_1'(t) = x_1(t) + 2x_2(t)$$
$$x_2'(t) = -2x_1(t) + 5x_2(t)$$

$A = \begin{bmatrix}1 & 2\\ -2 & 5\end{bmatrix}$

Find a general solution to this system. Then, consider the phase portrait that is shown above[1]. Is the phase portrait above a phase portrait for the given system? Justify your answer.

$$\begin{vmatrix}1-x & 2\\ -2 & 5-x\end{vmatrix} = x^2-6x+9 = (x-3)^2$$

$\lambda = 3, 3$

$\begin{bmatrix}-2 & 2\\ -2 & 2\end{bmatrix}$ $\quad v_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$(A-3I)\eta = v$ $\quad \left[\begin{array}{cc|c}-2 & 2 & 1\\ -2 & 2 & 1\end{array}\right] \Rightarrow \left[\begin{array}{cc|c}1 & -1 & -\frac{1}{2}\\ 0 & 0 & 0\end{array}\right]$

$\eta = \begin{pmatrix}-\frac{1}{2}\\ 0\end{pmatrix} + s\begin{pmatrix}1\\1\end{pmatrix}$

Choose $\eta = \begin{pmatrix}-\frac{1}{2}\\ 0\end{pmatrix}$

END OF EXAM

$$\boxed{\vec{x}(t) = \begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = C_1e^{3t}\begin{pmatrix}1\\1\end{pmatrix} + C_2\left(te^{3t}\begin{pmatrix}1\\1\end{pmatrix} + e^{3t}\begin{pmatrix}-\frac{1}{2}\\ 0\end{pmatrix}\right)}$$

[1]Figure generated using Matplotlib