Math 54 Final Exam (Practice 4)

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Name: ____

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Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Recall that P_4 is the set of polynomials with real coefficients of degree ≤ 4.

Part (a)

Let U be the set of polynomials in P_4 such that $p(0) = 0$ and $p(1) = -1$. Is U a subspace of P_4? If so, prove that U is a subspace, and find a basis for U. If not, give a proof that shows that U is not a subspace. [15 points]

Part (b)

Let V be the set of polynomials in P_4 such that $p(-2) = 0$ and $p(1) = 0$. Is U a subspace of P_4? If so, prove that V is a subspace, and find a basis for V. If not, give a proof that shows that V is not a subspace. [15 points]

Problem 2 (30 points)

Consider the map $T : P_2 \to M_{2\times 2}$ given by

$$T(p(x)) = \begin{bmatrix} p(-1) & p(1) \\ p'(-1) & p'(1) \end{bmatrix}$$

Part (a)

Prove that T is a linear transformation. [5 points]

Part (b)

Find a basis for $\ker(T)$ and a basis for $\text{range}(T)$. [25 points]

Problem 3 (30 points)

Consider the matrix

$$A = \begin{bmatrix} x & x+2 & 0 \\ x+2 & 2x+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Part (a)

Find all x such that the matrix A is positive definite. [15 points]

Part (b)

Is the matrix A positive semidefinite when $x = 0$? Justify your answer with a proof or a computation. [15 points]

Problem 4 (30 points)

Consider the subspace W of points $(x, y, z) \in \mathbb{R}^3$ such that $-3x + 2y - z = 0$.

Part (a)

Find an orthonormal basis for W. [10 points]

Part (b)

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(v) = \text{proj}_{W^\perp}(v)$$

Find the matrix $[T]_{\mathcal{B}\to\mathcal{B}}$ of T with respect to the standard basis $\mathcal{B}$ for $\mathbb{R}^3$.

Problem 5 (20 points)

Find the general solution to the following system.

$$x_1'(t) = x_1(t) + x_3(t)$$

$$x_2'(t) = x_2(t) + x_3(t)$$

$$x_3'(t) = x_1(t) + x_2(t) - x_3(t)$$

Problem 6 (40 points)

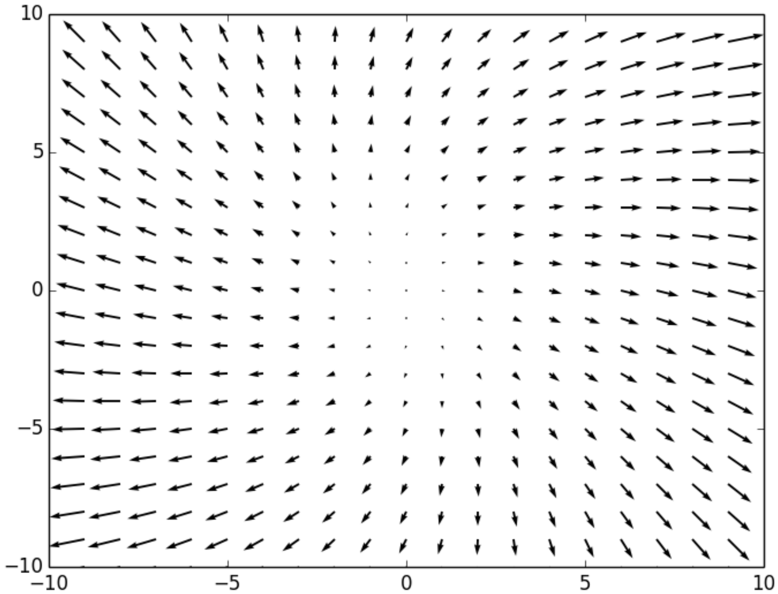
Find the Fourier series expansion for

$$f(\theta) = |\theta|, \quad -\pi < x < \pi$$

Use Plancherel's theorem to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = 1 + \frac{1}{81} + \frac{1}{625} + ...$$

(Hint: You should get $\widehat{f}(n) = \frac{\cos(n\pi)-1}{\pi n^2}$ for $n \neq 0$. Don't forget to also calculate $\widehat{f}(0)$.)



Problem 7 (20 points)

Consider the system of differential equations given by

$$x_1'(t) = x_1(t) + 2x_2(t)$$

$$x_2'(t) = -2x_1(t) + 5x_2(t)$$

Find a general solution to this system. Then, consider the phase portrait that is shown above[1]. Is the phase portrait above a phase portrait for the given system? Justify your answer.

END OF EXAM

[1] Figure generated using Matplotlib