Math 54 Final Exam (Practice 3)

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Name: Answer Key

SSID: ____________________

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Part (a)

Let U be the subset of 3 by 3 matrices with integer entries. Is U a subspace of $M_{3\times 3}$? Prove your answer. [10 points]

No, not closed under scalar multiplication.

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \notin U.$$

Part (b)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the translation map, $T(x, y, z) = (x + 1, y + 1, z + 1)$. Is T a linear transformation? Prove your answer. [10 points]

No. $T(0,0,0) = 0$ for any linear transformation T

(Since $T(0,0,0) = T(0\cdot(0,0,0)) = 0T(0,0,0) = 0$)

but here, $T(0,0,0) = (1,1,1)$.

Part (c)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix multiplication map given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}_{A}$$

Is T an isometry? Prove your answer. [10 points]

Yes. A is an orthogonal matrix.

$$AA^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So T is an isometry.

Problem 2 (30 points)

Part (a)

Show that there is no one-to-one linear transformation T from $M_{2\times 2}$ to P_2. [15 points]

$\dim(M_{2\times2}) = 4$, $\dim(P_2) = 3$.

Let $T: M_{2\times2} \to P_2$ be a linear transformation.

Then $\dim(\text{range}(T)) + \dim(\ker(T)) = \dim(M_{2\times2}) = 4$.

Since $\dim(P_2)=3$, $\dim(\text{range}(T)) \leq 3$. So

$\dim(\ker(T)) \geq 1$ so $\ker(T)$ is nontrivial.

So T is not one-to-one.

Part (b)

Is there a bijective linear transformation T from $M_{2\times 2}$ to P_3? Either prove that one does not exist, or find an example of one. [15 points]

Yes. Consider $T: M_{2\times2} \to P_3$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + bx + cx^2 + dx^3.$$

Problem 3 (30 points)

Consider the linear transformation $T : P_4 \to P_4$ given by

$$T(p(x)) = \frac{d^2}{dx^2}((x-1)p(x))$$

Find the trace, determinant, and characteristic polynomial of T. Find all eigenvalues and eigenvectors of T. Is T diagonalizable?

$\mathcal{B} = \{1, x, x^2, x^3, x^4\}$

$T(1) = \frac{d^2}{dx^2}(x-1) = 0 \quad T(x) = \frac{d^2}{dx^2}(x^2-x) = 2$

$T(x^2) = \frac{d^2}{dx^2}(x^3-x^2) = 6x-2 \quad T(x^3) = \frac{d^2}{dx^2}(x^4-x^3) = 12x^2-6x$

$T(x^4) = \frac{d^2}{dx^2}(x^5-x^4) = 20x^3-12x^2$

$$[T]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 6 & -6 & 0 \\ 0 & 0 & 0 & 12 & -12 \\ 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\det(T) = 0$ (upper triangular)

$\operatorname{tr}(T) = 0$

$\boxed{\text{char}_T(x) = -x^5}$

$\lambda = 0 \quad v = c_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{p(x) = c_1 \quad \lambda = 0}$

$\boxed{\text{T is not diagonalizable.}}$

Problem 4 (30 points)

Consider the subspace W of $\mathbb{R}^4$ spanned by $\underset{v_2}{(1,2,1,0)}$ and $\underset{v_1}{(0,1,1,1)}$.

Part (a)

Find an orthonormal basis for W. [10 points]

$$w_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

$$v_2 - \langle v_2, w_1\rangle w_1 = \begin{pmatrix}1\\2\\1\\0\end{pmatrix} - \frac{1}{3}(3)\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\\-1\end{pmatrix}$$

$$w_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}$$

$$\boxed{w_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\1\\1\\1\end{pmatrix},\quad w_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}}$$

Part (b)

Find an orthonormal basis for $W^\perp$. [20 points]

A vector $(x_1, x_2, x_3, x_4) \in W^\perp$ if

$$(x_1, x_2, x_3, x_4)\cdot(1,2,1,0) = 0$$
$$(x_1, x_2, x_3, x_4)\cdot(0,1,1,1) = 0 \quad (\text{def of } W^\perp)$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases}$$

$$\begin{bmatrix}1 & 2 & 1 & 0\\ 0 & 1 & 1 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 1 & 1\end{bmatrix}$$

$$\Rightarrow x_3\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} + x_4\begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$$

$$\text{Basis for } W^\perp = \begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$$

$$\boxed{w_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}}$$

$$v_2 - \langle v_2, w_1\rangle w_1 = \begin{pmatrix}2\\-1\\0\\1\end{pmatrix} - \frac{1}{3}(3)\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

$$\boxed{w_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}}$$

(Practice 4)

Problem 6 (30 points)

Problem 5 (30 points)

Find a particular solution to the differential equation

$$y'' - 3y' + 2y = 2xe^{2x} - x$$

$(+(-x))$

such that $y(0) = 1$ and $y'(0) = 0$.

$y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0 \quad (r-1)(r-2) = 0 \quad r = 1, 2$

$y_{hom} = C_1 e^x + C_2 e^{2x}$

$y'' - 3y' + 2y = 2xe^{2x}$

y_P Basic guess: $y_P = (Ax+B)Ce^{2x}$

$= ACxe^{2x} + BCe^{2x}$

$\Rightarrow Axe^{2x} + \underbrace{Be^{2x}}_{\text{in } y_h}$ X

$\Rightarrow y_P = Ax^2e^{2x} + Bxe^{2x}$ ✓

$= (Ax^2 + Bx)e^{2x}$

$y'' - 3y' + 2y = -x$

$y_P = Ax + B \quad y_P' = A \quad y_P'' = 0$

$0 - 3A + 2Ax + 2B = -x$

$2A = -1 \quad -3A + 2B = 0$

$A = -\frac{1}{2} \quad B = -\frac{3}{4}$

$y_P' = 2(Ax^2 + Bx)e^{2x} + (2Ax + B)e^{2x}$

$= (2Ax^2 + (2A + 2B)x + B)e^{2x}$

$y_P'' = 2(2Ax^2 + (2A+2B)x + B)e^{2x}$

$+ (4Ax + (2A+2B))e^{2x}$

$= (4Ax^2 + (8A+4B)x + (2A+4B))e^{2x}$

$\cancel{4Ax^2e^{2x}} + (8A+4B)xe^{2x} + (2A+4B)e^{2x}$

$\cancel{-6Ax^2e^{2x}} - (6A+6B)xe^{2x} - 3Be^{2x} + \cancel{2Ax^2e^{2x}} + 2Bxe^{2x} = 2xe^{2x}$

$2Axe^{2x} + (2A + B)e^{2x} = 2xe^{2x}$

$2A = 2 \quad 2A + B = 0 \quad A = 1, B = -2$

$y_P = x^2e^{2x} - 2xe^{2x} - \frac{1}{2}x - \frac{3}{4}$

$y = x^2e^{2x} - 2xe^{2x} - \frac{1}{2}x - \frac{3}{4}$

$+ C_1e^x + C_2e^{2x}$

$y' = 2x^2e^{2x} + 2xe^{2x} - 4xe^{2x} - 2e^{2x} - \frac{1}{2}$

$+ C_1e^x + 2C_2e^{2x}$

$y(0) = 1 \quad -\frac{3}{4} + C_1 + C_2 = 1$

$y'(0) = 0 \quad -2 - \frac{1}{2} + C_1 + 2C_2 = 0$

$\begin{cases} C_1 + C_2 = \frac{7}{4} \\ C_1 + 2C_2 = \frac{5}{2} \end{cases}$

$C_2 = \frac{3}{4} \quad C_1 = 1$

$\boxed{y = x^2e^{2x} - 2xe^{2x} - \frac{1}{2}x - \frac{3}{4} + e^x + \frac{3}{4}e^{2x}}$

Problem 6 (30 points)

Find the Fourier series expansion for

$$f(\theta) = -1, \quad -\pi < x < 0 \qquad\qquad f(\theta) = 1, \quad 0 < x < \pi$$

Write the expansion as both an infinite sum of complex exponentials, and also as an infinite sum of sines and cosines.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-in\theta}\, d\theta + \frac{1}{2\pi}\int_{0}^{\pi} e^{-in\theta}\, d\theta$$

So $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{0} -1\, d\theta + \frac{1}{2\pi}\int_{0}^{\pi} d\theta = 0.$

$n \neq 0$ $\quad \hat{f}(n) = \frac{1}{2\pi}\left(\frac{e^{-in\theta}}{in}\right)\Big|_{-\pi}^{0} + \frac{1}{2\pi}\left(\frac{e^{-in\theta}}{-in}\right)\Big|_{0}^{\pi}$

$$= \frac{1}{2\pi}\frac{1}{in}\left(1 - e^{in\pi}\right) + \frac{1}{2\pi}\left(\frac{-1}{in}\right)\left(e^{-in\pi} - 1\right)$$

$$= \frac{1}{\pi}\frac{1}{in} + \left(-\frac{1}{2\pi i n}\right)\left(e^{in\pi} + e^{-in\pi}\right)$$

$$= \frac{1}{\pi}\frac{1}{in} + \left(-\frac{1}{2\pi i n}\right)\underbrace{\left(\cos(n\pi) + \cancel{i\sin(n\pi)} + \cos(n\pi) - \cancel{i\sin(n\pi)}\right)}_{2(-1)^n}$$

$$= \frac{1}{\pi i n} + \frac{(-1)^{n+1}}{\pi i n} = \frac{1}{\pi i n}\left(1 + (-1)^{n+1}\right)$$

So $\boxed{f(\theta) \sim \sum_{\substack{n \neq 0 \\ n \in \mathbb{Z}}} \frac{1 + (-1)^{n+1}}{\pi i n} e^{in\theta}}$

$$= \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{\pi i n} e^{in\theta} + \sum_{n=1}^{\infty} \frac{1 + (-1)^{-n+1}}{-\pi i n} e^{-in\theta}$$

$$= \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{\pi i n}\left(\cos(n\theta) + i\sin(n\theta)\right) + \sum_{n=1}^{\infty} -\frac{1 + (-1)^{n+1}}{\pi i n}\left(\cos(n\theta) - i\sin(n\theta)\right)$$

$$= \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{\pi \cancel{i} n}\left(2\cancel{i}\sin(n\theta)\right) = \boxed{\sum_{n=1}^{\infty} \frac{2\left(1 + (-1)^{n+1}\right)}{\pi n}\sin(n\theta)}$$

Problem 7 (20 points)

Match the systems of differential equations to the appropriate phase portrait **on the next page**. Note that not every figure on the next page will be used[1].

$\boxed{D}$
$$x_1'(t) = x_2(t) \qquad x_2'(t) = -x_1(t) \tag{1}$$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\begin{vmatrix} -x & 1 \\ -1 & -x \end{vmatrix} = x^2+1$

$\lambda = 0+i, 0-i$ ellipses

$\boxed{E}$
$$x_1'(t) = 2x_1(t) + x_2(t) \qquad x_2'(t) = -x_1(t) + 2x_2(t) \tag{2}$$

$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ $\begin{vmatrix} 2-x & 1 \\ -1 & 2-x \end{vmatrix} = x^2-4x+5$

$\lambda = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$ unstable spiral

$\boxed{A}$
$$x_1'(t) = -x_1(t) \qquad x_2'(t) = -x_2(t) \tag{3}$$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ stable, every vector is an eigenvector

$\boxed{H}$
$$x_1'(t) = x_1(t) + 2x_2(t) \qquad x_2'(t) = 2x_1(t) + x_2(t) \tag{4}$$

$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\begin{vmatrix} 1-x & 2 \\ 2 & 1-x \end{vmatrix} = x^2-2x-3$
$= (x-3)(x+1)$

$\lambda = 3, -1$
saddle pt

$\boxed{C}$
$$x_1'(t) = -3x_1(t) + x_2(t) \qquad x_2'(t) = -x_1(t) - 3x_2(t) \tag{5}$$

$\begin{bmatrix} -3 & 1 \\ -1 & -3 \end{bmatrix}$ $\begin{vmatrix} -3-x & 1 \\ -1 & -3-x \end{vmatrix} = x^2+6x+10$

$\lambda = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i$
stable spiral

$\boxed{G}$
$$x_1'(t) = -x_1(t) + x_2(t) \qquad x_2'(t) = -x_1(t) - 3x_2(t) \tag{6}$$

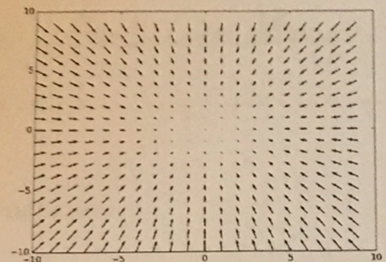
$\begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$ $\begin{vmatrix} -1-x & 1 \\ -1 & -3-x \end{vmatrix}$

$= x^2+4x+4$
$= (x+2)^2$ repeated
$\lambda = -2$ stable

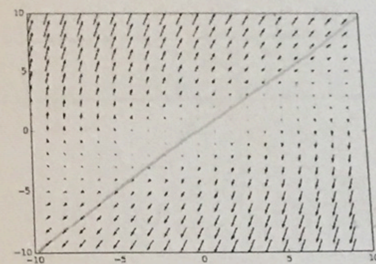
END OF EXAM

(Figures on next page)

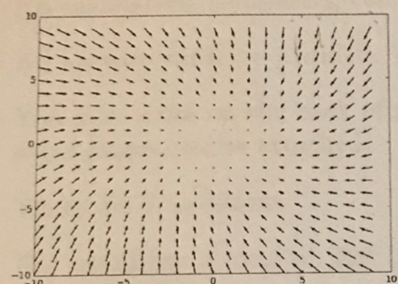
[1] Figures generated using Matplotlib



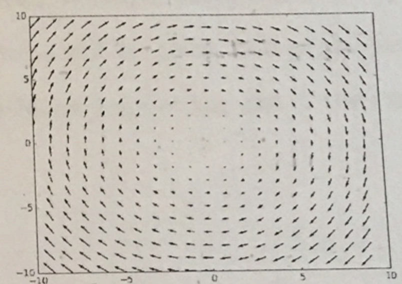
(a) Graph (A)



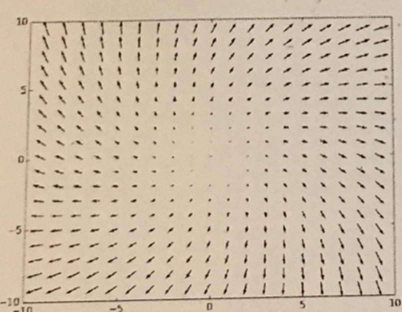
(b) Graph (B)



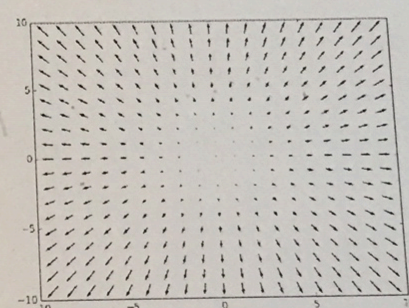
(c) Graph (C)



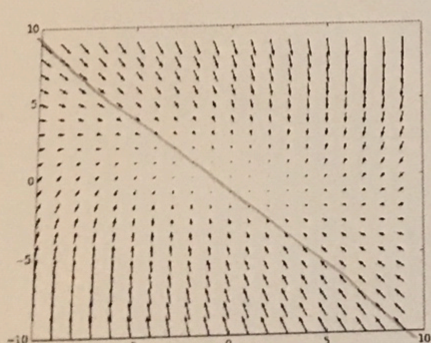
(d) Graph (D)



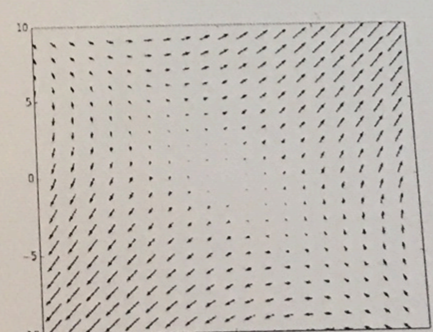
(e) Graph (E)



(f) Graph (F)



(g) Graph (G)



(h) Graph (H)