Math 54 Final Exam (Practice 2)

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Name: ____

SSID: ____

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Let S be the set of 2 by 2 matrices that commute with the matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$.

Part (a)

Show that S is a subspace of $M_{2\times 2}$. [10 points]

Part (b)

Find a basis for S. [20 points]

Problem 2 (30 points)

Consider

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \right\}$$

Part (a)

Show that $\mathcal{B}$ is a basis for the vector space of skew-symmetric 3 by 3 matrices, $\text{Skew}_{3\times 3}$. [10 points]

Part (b)

For $t : \text{Skew}_{3\times 3} \to \text{Skew}_{3\times 3}$, calculate $[t]_{\mathcal{B}\to\mathcal{B}}$, where t is the matrix transpose linear transformation that sends A to A^t. [20 points]

Problem 3 (30 points)

Find all least squares solutions to $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 3 & 1 \\ 1 & 1 & 0 & -1 & -1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Problem 4 (30 points)

Consider the matrix multiplication linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$, given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Part (a)

Find a basis for $\ker(T)$ and $\mathrm{range}(T)$. [10 points]

Part (b)

Determine if the point $(1, 3, 4, 5)$ is in $\mathrm{range}(T)$. If not, find the distance from $(1, 3, 4, 5)$ to $\mathrm{range}(T)$ in $\mathbb{R}^4$. [20 points]

Problem 5 (20 points)

For each system, find a general solution. [10 points each]

$$x_1'(t) = -2x_1(t) + 3x_2(t)$$
$$x_2'(t) = -2x_2(t)$$

$$x_1'(t) = -x_1(t) - x_2(t)$$
$$x_2'(t) = 2x_1(t) - 3x_2(t)$$

Problem 6 (40 points)

Calculate the Fourier series expansion for the function

$$f(\theta) = \theta \qquad -\pi < \theta < \pi$$

Write your answer both as a sum of complex exponentials, and as a sum of sines and cosines. Use the expansion and Plancherel's theorem to find the value of the convergent infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Problem 7 (20 points)

Consider the following six second order differential equations.

$$y'' + 2y' + 17y = 0 \tag{1}$$

$$y'' - 3y' + 2y = 2x \tag{2}$$

$$y'' - 2y' + 17y = 0 \tag{3}$$

$$y'' + 2y' + y = 0 \tag{4}$$

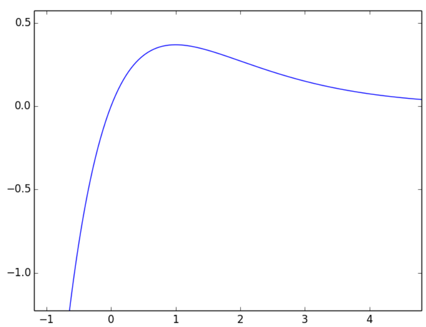
$$y'' + 3y' + 2y = 0 \tag{5}$$

Match these differential equations to a graph of one of their particular solutions **on the next page**, which shows all of the figures[1]. Note that every equation here will be matched to a different graph, but not all graphs on the next page will be used.

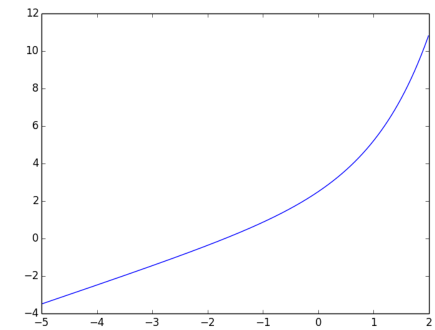
END OF EXAM

(Figures on next page)

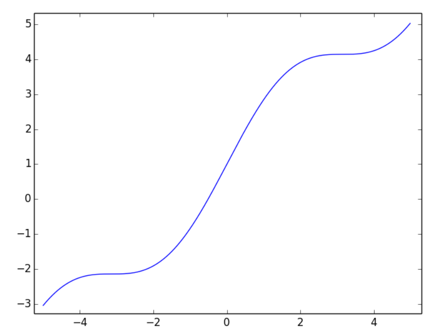
[1]Figures generated using Matplotlib



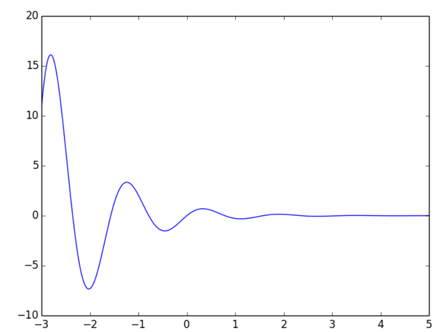
(a) Graph (A)



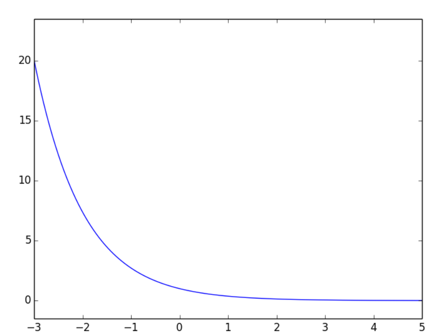
(b) Graph (B)



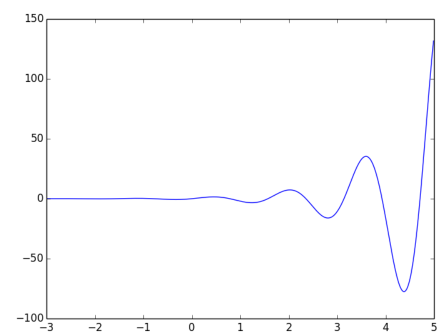
(c) Graph (C)



(d) Graph (D)



(e) Graph (E)



(f) Graph (F)