Math 54 Final Exam (Practice 1)

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Name: Answer Key

SSID:

Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Let U be the set of symmetric 3 by 3 matrices with trace equal to zero. Show that U is a subspace of $M_{3\times 3}$, and find (with proof) a basis for U.

Let A, B be symmetric 3×3 matrices with trace 0.

$c_1A + c_2B$ is also symmetric, and

$$tr(c_1A+c_2B) = c_1 tr(A) + c_2 tr(B)$$
$$= c_1\cdot 0 + c_2 \cdot 0 = 0.$$

So c_1A+c_2B is also symmetric with trace 0.

So U is a subspace of $M_{3\times 3}$.

A general matrix in U looks like

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{bmatrix}$$

$$= a\begin{bmatrix} 1&0&0\\0&0&0\\0&0&-1\end{bmatrix} + b\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0\end{bmatrix} + c\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0\end{bmatrix} + d\begin{bmatrix} 0&0&0\\0&1&0\\0&0&-1\end{bmatrix} + e\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

So consider $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 1&0&0\\0&0&0\\0&0&-1\end{bmatrix}}_{M_1}, \underbrace{\begin{bmatrix} 0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{M_2}, \underbrace{\begin{bmatrix} 0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{M_3}, \underbrace{\begin{bmatrix} 0&0&0\\0&1&0\\0&0&-1\end{bmatrix}}_{M_4}, \underbrace{\begin{bmatrix} 0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{M_5} \right\}$

The above equation shows $\mathcal{B}$ spans U.

$\mathcal{B}$ is also linearly independent.

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 + c_5M_5 = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_2 & c_4 & c_5 \\ c_3 & c_5 & -c_1-c_4 \end{bmatrix} = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$c_1 = c_2 = c_3 = c_4 = c_5 = 0.$

So $\mathcal{B}$ is a basis for U.

Problem 2 (30 points)

Consider the linear transformation $T : \mathbb{C}^2 \to \mathbb{R}^4$ given by

$$T(a+bi, c+di) = (a+c, a+d, \overset{b-c}{\cancel{b+c}}, b+d)$$

Part (a)

Show that T is bijective. [15 points]

$\ker(T)$: $\begin{matrix} a+c=0 \\ a+d=0 \\ b-c=0 \\ b+d=0 \end{matrix}$ $\quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightarrow \det = 1(1)(1) - 1(-1)(1) = 2 \neq 0$

$a=b=c=d=0$. So $\ker(T) = \{0\}$.

Since $\dim(\mathbb{C}^2) = \dim(\mathbb{R}^4) = 4$ and $\text{nullity}(T) = 0$,

$\text{nullity}(T) + \text{rank}(T) = \dim(\mathbb{C}^2)$

$\Rightarrow \text{rank}(T) = 4 = \dim(\text{range}(T)) = 4$.

So $\text{range}(T) = \mathbb{R}^4$. (So T is onto.) Since T is 1-1 and onto, T is bijective.

Part (b)

Find a formula for $T^{-1}(w, x, y, z)$, where $(w, x, y, z) \in \mathbb{R}^4$. [15 points]

$$\begin{aligned} a+c &= w \\ a+d &= x \\ b-c &= y \\ b+d &= z \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & w \\ 1 & 0 & 0 & 1 & x \\ 0 & 1 & -1 & 0 & y \\ 0 & 1 & 0 & 1 & z \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & w \\ 0 & 0 & 1 & -1 & w-x \\ 0 & 1 & -1 & 0 & y \\ 0 & 0 & 1 & 1 & -y+z \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & w \\ 0 & 1 & -1 & 0 & y \\ 0 & 0 & 1 & 1 & -y+z \\ 0 & 0 & 0 & 1 & \frac{1}{2}(-w+x-y+z) \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & w \\ 0 & 1 & -1 & 0 & y \\ 0 & 0 & 1 & 0 & \frac{1}{2}w-\frac{1}{2}x-\frac{1}{2}y+\frac{1}{2}z \\ 0 & 0 & 0 & 1 & -\frac{1}{2}w+\frac{1}{2}x-\frac{1}{2}y+\frac{1}{2}z \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{2}w+\frac{1}{2}x+\frac{1}{2}y-\frac{1}{2}z \\ 0 & 1 & 0 & 0 & \frac{1}{2}w-\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z \\ 0 & 0 & 1 & 0 & \frac{1}{2}w-\frac{1}{2}x-\frac{1}{2}y+\frac{1}{2}z \\ 0 & 0 & 0 & 1 & -\frac{1}{2}w+\frac{1}{2}x-\frac{1}{2}y+\frac{1}{2}z \end{array}\right]$$

$$T^{-1}(w,x,y,z) = \left(\left(\tfrac{1}{2}w+\tfrac{1}{2}x+\tfrac{1}{2}y-\tfrac{1}{2}z\right) + \left(\tfrac{1}{2}w-\tfrac{1}{2}x+\tfrac{1}{2}y+\tfrac{1}{2}z\right)i, \left(\tfrac{1}{2}w-\tfrac{1}{2}x-\tfrac{1}{2}y+\tfrac{1}{2}z\right) + \left(-\tfrac{1}{2}w+\tfrac{1}{2}x-\tfrac{1}{2}y+\tfrac{1}{2}z\right)i \right)$$

Problem 3 (30 points)

Let $t : M_{2\times 2} \to M_{2\times 2}$ denote the matrix transpose map that sends A to A^t.

Part (a)

Show that t is a linear transformation. [5 points]

$$
\begin{aligned}
&t(c_1 M_1 + c_2 M_2) \\
&= (c_1 M_1 + c_2 M_2)^t = (c_1 M_1)^t + (c_2 M_2)^t \\
&\qquad = c_1 M_1^t + c_2 M_2^t \\
&\qquad = c_1 t(M_1) + c_2 t(M_2)
\end{aligned}
$$

Part (b)

Find the trace, determinant, and characteristic polynomial of t, and find all eigenvalues and eigenvectors. Is t diagonalizable? [25 points]

Choose $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$t\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad t\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$t\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad t\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

So $[t]_{\mathcal{B}\to\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad \boxed{tr(T) = 2}$

$det(T) = 1(1)(-1)$

$\boxed{det(T) = -1}$

$$
\begin{vmatrix} 1-x & 0 & 0 & 0 \\ 0 & -x & 1 & 0 \\ 0 & 1 & -x & 0 \\ 0 & 0 & 0 & 1-x \end{vmatrix} = (1-x)(1-x)(x^2-1)
$$

$$
= \boxed{(x-1)^3(x+1) = char_T(x)}
$$

$\lambda = 1$

$$
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = -1$

$$
\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}
$$

$c_1 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

$$
\boxed{\begin{gathered} \lambda = 1 \\ c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \lambda = -1 \\ c_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{gathered}}
$$

$\boxed{t \text{ is diagonalizable}}$

Problem 4 (30 points)

Let W be the plane $x - y - 4z = 0$ in $\mathbb{R}^3$. Which of the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ is closest to W? Justify your answer with calculations or a proof.

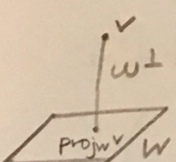
Basis for $x-y-4z=0$: $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}4\\0\\1\end{pmatrix}$

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \Rightarrow w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$v_2 - \langle v_2, w_1\rangle w_1 = \begin{pmatrix}4\\0\\1\end{pmatrix} - \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(4)\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\Rightarrow w_2 = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$, $w_2 = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is an ONB for W.



$\text{dist} = \|w^\perp\| = \|v - \text{proj}_W v\|$

$$\text{proj}_W(1,0,0) = \frac{1}{\sqrt{2}}w_1 + \frac{2}{3}w_2 = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{pmatrix} + \begin{pmatrix}\frac{4}{9}\\-\frac{4}{9}\\\frac{2}{9}\end{pmatrix}$$

$w^\perp = w - \text{proj}_W(1,0,0)$
$= (1,0,0) - (\frac{17}{18}, \frac{1}{18}, \frac{2}{9})$
$= (\frac{1}{18}, -\frac{1}{18}, -\frac{2}{9})$
$\text{dist} = \|w^\perp\|$

$= \begin{pmatrix}\frac{17}{18}\\\frac{1}{18}\\\frac{2}{9}\end{pmatrix}$ $\longrightarrow$ $\text{dist} = \sqrt{\left(\frac{1}{18}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{2}{9}\right)^2}$

smallest

$$\text{proj}_W(0,1,0) = \frac{1}{\sqrt{2}}w_1 - \frac{2}{3}w_2 = \begin{pmatrix}\frac{1}{18}\\\frac{17}{18}\\-\frac{2}{9}\end{pmatrix}$$

$w^\perp = w - \text{proj}_W(0,1,0)$
$= (0,1,0) - (\frac{1}{18}, \frac{17}{18}, -\frac{2}{9})$ $\longrightarrow$ $\text{dist} = \sqrt{\left(\frac{1}{18}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{2}{9}\right)^2}$
$= (-\frac{1}{18}, \frac{1}{18}, \frac{2}{9})$ $\text{dist} = \|w^\perp\|$

$$\text{proj}_W(0,0,1) = 0w_1 + \frac{1}{3}w_2 = \begin{pmatrix}\frac{2}{9}\\-\frac{2}{9}\\\frac{1}{9}\end{pmatrix}$$

$w^\perp = w - \text{proj}_W(0,0,1)$
$= (0,0,1) - (\frac{2}{9}, -\frac{2}{9}, \frac{1}{9})$ $\longrightarrow$ $\text{dist} = \sqrt{\left(\frac{2}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + \left(\frac{8}{9}\right)^2}$
$= (-\frac{2}{9}, \frac{2}{9}, \frac{8}{9})$ $\text{dist} = \|w^\perp\|$

(1, 0, 0) and (0,1,0) are closest to W
(same distance away)

Problem 5 (30 points)

Solve the following homgeneous second order linear differential equation in two ways: (1) by converting it to a first order system and solving it, and (2) by using second-order differential equation methods.

$$y'' - y' + y = 0 \qquad y'' = y' - y$$

(1) $\vec{x} = \begin{bmatrix} y \\ y' \end{bmatrix}$ $\quad \vec{x}\,' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -y+y' \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad \begin{vmatrix} -\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix}$

$$= -\lambda(1-\lambda) + 1 = \lambda^2 - \lambda + 1$$

$$\lambda = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$A - \left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) I = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2} i & 1 \\ -1 & \frac{1}{2} - \frac{\sqrt{3}}{2} i \end{bmatrix}$ $\quad \vec{v} = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{3}}{2} i \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 - \sqrt{3} i \\ 2 \end{pmatrix}$$

$$e^{\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)x} \begin{pmatrix} 1 - \sqrt{3} i \\ 2 \end{pmatrix}$$

$$= e^{\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2} x\right) + i \sin\left(\frac{\sqrt{3}}{2} x\right)\right) \begin{pmatrix} 1 - \sqrt{3} i \\ 2 \end{pmatrix}$$

$$= e^{\frac{1}{2}x} \begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2} x\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2} x\right) \\ 2\cos\left(\frac{\sqrt{3}}{2} x\right) \end{pmatrix} + i e^{\frac{1}{2}x} \begin{pmatrix} \sin\left(\frac{\sqrt{3}}{2} x\right) - \sqrt{3} \cos\left(\frac{\sqrt{3}}{2} x\right) \\ 2 \sin\left(\frac{\sqrt{3}}{2} x\right) \end{pmatrix}$$

$$\begin{bmatrix} y \\ y' \end{bmatrix} = C_1 e^{\frac{1}{2}x} \begin{pmatrix} \cos\left(\frac{\sqrt{3}}{2} x\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2} x\right) \\ 2\cos\left(\frac{\sqrt{3}}{2} x\right) \end{pmatrix} + C_2 e^{\frac{1}{2}x} \begin{pmatrix} \sin\left(\frac{\sqrt{3}}{2} x\right) - \sqrt{3} \cos\left(\frac{\sqrt{3}}{2} x\right) \\ 2 \sin\left(\frac{\sqrt{3}}{2} x\right) \end{pmatrix}$$

$$\boxed{y(x) = C_1 e^{\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2} x\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2} x\right)\right) + C_2 e^{\frac{1}{2}x}\left(\sin\left(\frac{\sqrt{3}}{2} x\right) - \sqrt{3} \cos\left(\frac{\sqrt{3}}{2} x\right)\right)}$$

(2) Guess $y = e^{rx}$. $\quad r^2 e^{rx} - r e^{rx} + e^{rx} = 0$

$$r^2 - r + 1 = 0$$

$$r = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\boxed{y(x) = C_1 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2} x\right) + C_2 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2} x\right)}$$

Math 54 Final Exam (Practice 4)

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Problem 6 (30 points)

Calculate the Fourier series expansion for the function

$$f(\theta) = e^{\theta} \qquad -\pi < \theta < \pi$$

Write your answer both as a sum of complex exponentials, and as a sum of sines and cosines.

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta$$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\theta}\,d\theta = \frac{e^{\pi}-e^{-\pi}}{2\pi}$$

$n \neq 0$: $$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\theta}e^{-in\theta}\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)\theta}\,d\theta$$

$$= \frac{1}{2\pi}\left(\frac{e^{(1-in)\theta}}{1-in}\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{e^{(1-in)\pi}}{1-in} - \frac{e^{-(1-in)\pi}}{1-in}\right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{1-in}\right)\left(e^{\pi}\underbrace{e^{-in\pi}}_{\cos(n\pi)-i\sin(n\pi)} - e^{-\pi}\underbrace{e^{in\pi}}_{\cos(n\pi)+i\sin(n\pi)}\right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{1-in}\right)\left(e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right)$$

$$= \frac{(-1)^n}{2\pi}\,\frac{1}{1-in}\left(e^{\pi}-e^{-\pi}\right)$$

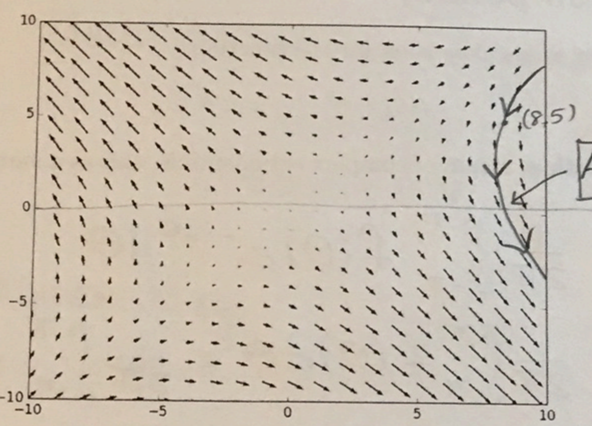
$$\boxed{f(\theta) \sim \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n\neq 0,\, n\in\mathbb{Z}} \frac{(-1)^n}{2\pi}\,\frac{1}{1-in}\left(e^{\pi}-e^{-\pi}\right)e^{in\theta}}$$

$$= \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2\pi}\,\frac{1}{1-in}\left(e^{\pi}-e^{-\pi}\right)e^{in\theta} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2\pi}\,\underbrace{\frac{1}{1+in}}_{n\to -n}\left(e^{\pi}-e^{-\pi}\right)e^{-in\theta}$$

$$= \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2\pi}\,\frac{1}{1-in}\left(e^{\pi}-e^{-\pi}\right)\left(\cos(n\theta)+i\sin(n\theta)\right)$$
$$+ \sum_{n=1}^{\infty}\frac{(-1)^n}{2\pi}\,\frac{1}{1+in}\left(e^{\pi}-e^{-\pi}\right)\left(\cos(n\theta)-i\sin(n\theta)\right)$$

$$\frac{1}{1-in} + \frac{1}{1+in} = \frac{2}{1+n^2} \qquad i\left(\frac{1}{1-in} - \frac{1}{1+in}\right) = i\left(\frac{2in}{1+n^2}\right) = -\frac{2n}{1+n^2}$$

$$= \boxed{\frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2\pi}\left(e^{\pi}-e^{-\pi}\right)\frac{2}{1+n^2}\cos(n\theta) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2\pi}\left(e^{\pi}-e^{-\pi}\right)\frac{2n}{1+n^2}\sin(n\theta)}$$



Animal B dies out first

Problem 7 (20 points)

Consider the system of differential equations

$$x_1'(t) = x_1(t) - 2x_2(t)$$
$$x_2'(t) = -2x_1(t) + x_2(t)$$

Suppose that $x_1(t)$ and $x_2(t)$ represent the populations of two animals, A and B, in thousands. The phase portrait is shown above[1].

- Find the general solution for the system above.
- Graph the trajectory on the phase portrait for the initial condition $x_1(0) = 8$, $x_2(0) = 5$. In this case, which animal dies out first, Animal A or Animal B? Justify your answer.
- Is $(0, 0)$ a stable equilibrium, an unstable equilibrium, or neither?

$$\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \to \begin{vmatrix} 1-x & -2 \\ -2 & 1-x \end{vmatrix} = (1-x)^2 - 4 = x^2 - 2x - 3$$
$$= (x-3)(x+1)$$

$$\lambda = 3, \quad \lambda = -1$$

$$\lambda = 3: \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \lambda = -1: \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

(0,0) is neither stable nor unstable (saddle point)

END OF EXAM

[1]Generated using Matplotlib