Math 54 Final Exam (Practice 1)

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Name: ____

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Instructions:

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

Problem 1 (30 points)

Let U be the set of symmetric 3 by 3 matrices with trace equal to zero. Show that U is a subspace of $M_{3\times 3}$, and find (with proof) a basis for U.

Problem 2 (30 points)

Consider the linear transformation $T : \mathbb{C}^2 \to \mathbb{R}^4$ given by

$$T(a + bi, c + di) = (a + c, a + d, b - c, b + d)$$

Part (a)

Show that T is bijective. [15 points]

Part (b)

Find a formula for $T^{-1}(w, x, y, z)$, where $(w, x, y, z) \in \mathbb{R}^4$. [15 points]

Problem 3 (30 points)

Let $t : M_{2\times 2} \to M_{2\times 2}$ denote the matrix transpose map that sends A to A^t.

Part (a)

Show that t is a linear transformation. [5 points]

Part (b)

Find the trace, determinant, and characteristic polynomial of t, and find all eigenvalues and eigenvectors. Is t diagonalizable? [25 points]

Problem 4 (30 points)

Let W be the plane $x - y - 4z = 0$ in $\mathbb{R}^3$. Which of the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ is closest to W? Justify your answer with calculations or a proof.

Problem 5 (30 points)

Solve the following homgeneous second order linear differential equation in two ways: (1) by converting it to a first order system and solving it, and (2) by using second-order differential equation methods.

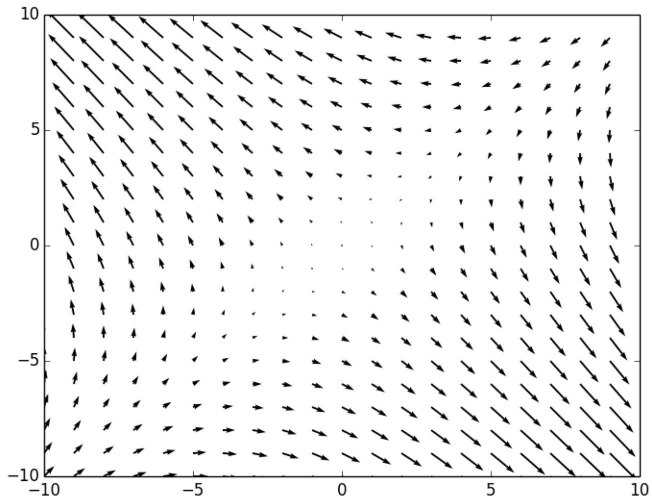
$$y'' - y' + y = 0$$

Problem 6 (30 points)

Calculate the Fourier series expansion for the function

$$f(\theta) = e^{\theta} \qquad -\pi < \theta < \pi$$

Write your answer both as a sum of complex exponentials, and as a sum of sines and cosines.



Problem 7 (20 points)

Consider the system of differential equations

$$x_1'(t) = x_1(t) - 2x_2(t)$$
$$x_2'(t) = -2x_1(t) + x_2(t)$$

Suppose that $x_1(t)$ and $x_2(t)$ represent the populations of two animals, A and B, in thousands. The phase portrait is shown above[1].

- Find the general solution for the system above.
- Graph the trajectory on the phase portrait for the initial condition $x_1(0) = 8$, $x_2(0) = 5$. In this case, which animal dies out first, Animal A or Animal B? Justify your answer.
- Is $(0, 0)$ a stable equilibrium, an unstable equilibrium, or neither?

END OF EXAM

[1] Generated using Matplotlib