00038

# Math 54 Final Exam

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Name: Answer Key

SSID: ____________________

**Instructions:**

- This exam is **110 minutes** long.
- No calculators, computers, cell phones, textbooks, notes, or cheat sheets are allowed.
- All answers must be justified. Unjustified answers will be given little or no credit.
- You may write on the back of pages or on the blank page at the end of the exam. No extra pages can be attached.
- There are 7 questions.
- The exam has a total of **200 points**.
- Good luck!

## Problem 1 (30 points)

Let $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Consider the maps $S : M_{2\times 2} \to M_{2\times 2}$ given by

$$S(A) = A + B$$

and $T : M_{2\times 2} \to M_{2\times 2}$ given by

$$T(A) = BA$$

### Part (a)

Show that $T$ is a linear transformation. Determine (with proof) whether or not $S$ is a linear transformation. [10 points]

Check linearity: $T(c_1 A_1 + c_2 A_2)$

$$= B(c_1 A_1 + c_2 A_2)$$
$$= c_1(BA_1) + c_2(BA_2)$$
$$= c_1 T(A_1) + c_2 T(A_2) \checkmark$$

So $T$ is a linear transformation.

$S$ is not a linear transformation since

$$S\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$\neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

## Part (b)

Find a basis for ker($T$). While you must show your calculations, you don't need to prove that the set you give is a basis. [15 points]

$$T(A) = BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 & x_2 - x_4 \\ -x_1 + x_3 & -x_2 + x_4 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_3 &= 0 \\ -x_1 + x_3 &= 0 \\ x_2 - x_4 &= 0 \\ -x_2 + x_4 &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}}$$

## Part (c)

Let $U$ be the subspace of 2 by 2 matrices that can be written as $BA$ for some 2 by 2 matrix $A$, where $B$ is the matrix above. What is the dimension of $U$? [5 points]

$\dim(\ker(T)) = 2$ by part (b).
← nullity(T)

$U = \text{range}(T)$.

Use the Rank-Nullity Theorem.

$$\text{rank}(T) + \text{nullity}(T) = \dim(M_{2\times 2})$$

$$\text{rank}(T) + 2 = 4$$

$$\text{rank}(T) = 2 \quad \text{so } \boxed{\dim(U) = 2}$$

## Problem 2 (30 points)

Consider the linear transformation $T : P_2 \to \mathbb{R}^3$ given by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \end{bmatrix}$$

### Part (a)

Show that $T$ is bijective. [15 points]

$$T(a + bx + cx^2) = \begin{bmatrix} a \\ a+b+c \\ a+2b+4c \end{bmatrix}$$

ker(T): $a = 0$

$a+b+c = 0 \Rightarrow b+c=0 \Rightarrow 2b+2c=0$

$a+2b+4c=0 \Rightarrow 2b+4c=0 \Longrightarrow 2c=0 \quad c=0$

$\Rightarrow b=0$

So $\ker(T) = \{0 + 0x + 0x^2\}$

so $T$ is one-to-one.

By Rank-Nullity, $\text{rank}(T) = \dim(P_2) - \text{nullity}(T)$

$= 3 - 0 = 3$

so $\text{range}(T) = \mathbb{R}^3$. So $T$ is onto also.

$\Rightarrow T$ is bijective.

### Part (b)

Find a formula for $T^{-1}(u, v, w)$, where $(u, v, w) \in \mathbb{R}^3$. [15 points]

$$\begin{bmatrix} a \\ a+b+c \\ a+2b+4c \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$a = u$

$u + b + c = v \Rightarrow b + c = v - u$

$u + 2b + 4c = w \Rightarrow 2b + 4c = w - u$

$b + 2c = \frac{1}{2}(w - u)$

$c = \frac{1}{2}w - v + \frac{1}{2}u$

$b = (v-u) - (\frac{1}{2}w - v + \frac{1}{2}u)$

$= -\frac{1}{2}w + 2v - \frac{3}{2}u$

$$\boxed{T^{-1}(u,v,w) = u + \left(-\frac{3}{2}u + 2v - \frac{1}{2}w\right)x + \left(\frac{1}{2}u - v + \frac{1}{2}w\right)x^2}$$

## Problem 3 (30 points)

Let $W$ be the subspace of points $(x, y, z)$ in $\mathbb{R}^3$ satisfying $x - y + z = 0$.

### Part (a)

Find an orthonormal basis for $W$. [10 points]

Basis for W: $\underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$$\boxed{\begin{aligned} w_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ w_2 &= \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \end{aligned}}$$

$$v_2 - \langle v_2, w_1 \rangle w_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}(-1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

basis (ONB) for W

### Part (b)

$W$ is a plane in $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the map that reflects a point $(x, y, z)$ across the plane represented by $W$. Explicitly, this map is given by

$$T(x, y, z) = \text{proj}_W(x, y, z) - \text{proj}_{W^\perp}(x, y, z)$$

Find $[T]_{\mathcal{B}\to\mathcal{B}}$ where $\mathcal{B}$ is the standard basis for $\mathbb{R}^3$. [20 points]

(Hint: You may use that $W^\perp$ is spanned by $(1, -1, 1)$, but be sure to orthonormalize this vector if you want to use this fact. This problem can actually be done without this fact too, if you want.)

$w_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ basis (ONB) for $W^\perp$

$$\text{proj}_W(1,0,0) = \frac{1}{2}(1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left(-\frac{1}{6}\right)\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$\text{proj}_{W^\perp}(1,0,0) = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad T(1,0,0) = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$$

$$\text{proj}_W(0,1,0) = \frac{1}{2}(1)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left(\frac{1}{6}\right)\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\text{proj}_{W^\perp}(0,1,0) = -\frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} \quad T(0,1,0) = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\text{proj}_W(0,0,1) = 0w_1 + \frac{2}{6}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$\text{proj}_{W^\perp}(0,0,1) = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad T(0,0,1) = \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\boxed{[T]_{\mathcal{B}\to\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}}$$

## Problem 4 (25 points)

Consider the matrix
$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & -2 & -1 & -3 \\ -1 & -2 & -1 & -1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 3 & 0 & -1 & 1 \end{bmatrix}$$

### Part (a)

Find an orthonormal basis for Col$(A)$. [10 points]

$$\begin{bmatrix} 1 & 2 & 1 & 1 & -2 & -1 & -3 \\ -1 & -2 & -1 & -1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 3 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 1 & -2 & -1 & -3 \\ 0 & -1 & 1 & 3 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis: $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ $\quad w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$v_2 - \langle v_2, w_1 \rangle w_1 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} - \frac{1}{2}(4)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\boxed{w_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad w_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}$$

### Part (b)

Let $b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. Find all least square solutions to $Ax = b$. [15 points]

$$A\hat{x} = \text{proj}_{Col(A)}\, b$$

$$\text{proj}_{Col(A)}\, b = \frac{1}{2}(-2)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{ccccccc|c} 1 & 2 & 1 & 1 & -2 & -1 & -3 & -1 \\ -1 & -2 & -1 & -1 & 2 & 1 & 3 & 1 \\ 0 & -1 & 1 & 3 & 0 & -1 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccccccc|c} 1 & 0 & 3 & 7 & -2 & -3 & -1 & 1 \\ 0 & 1 & -1 & -3 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \\ -R_3 \\ R_1 + R_2 \end{array}$$

$$\hat{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -7 \\ 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_6\begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_7\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

## Problem 5 (24 points)

### Part (a)

Find a general solution to the following second order differential equation. [12 points]

$$y'' - y = 2xe^x$$

$r^2 - 1 = 0 \quad (r-1)(r+1) = 0 \quad r = -1, 1$

$y_h = C_1 e^x + C_2 e^{-x}$

Basic guess: $y_p = (Ax+B)Ce^x = ACxe^x + BCe^x$

$\Rightarrow (Ax+B)e^x$ but $Be^x$ in $y_h$

$\Downarrow$

$y_p = (Ax^2 + Bx)e^x$ ✓

$y_p' = (Ax^2 + (2A+B)x + B)e^x$

$y_p'' = (Ax^2 + (4A+B)x + (2A+2B))e^x$

$(Ax^2 + (4A+B)x + (2A+2B))e^x - (Ax^2+Bx)e^x$

$= 4Axe^x + (2A+2B)e^x = 2xe^x$

$4A = 2$

$2A + 2B = 0$

$A = \frac{1}{2} \quad B = -\frac{1}{2}$

$$\boxed{y = \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x\right)e^x + C_1e^x + C_2e^{-x}}$$

### Part (b)

Find a general solution to the following system of differential equations. [12 points]

$$x_1'(t) = -x_2(t)$$

$$x_2'(t) = 5x_1(t) + 2x_2(t)$$

$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$\vec{x}' = \begin{bmatrix} 0 & -1 \\ 5 & 2 \end{bmatrix} \vec{x}$

$\begin{vmatrix} -\lambda & -1 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 \qquad \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

$\lambda = 1+2i \qquad A - \lambda I = \begin{bmatrix} -1-2i & -1 \\ 5 & 1-2i \end{bmatrix} \qquad v = \begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$

$e^{(1+2i)t}\begin{pmatrix} 1 \\ -1-2i \end{pmatrix} = e^t(\cos(2t) + i\sin(2t))\begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$

$= \begin{pmatrix} e^t\cos(2t) \\ -e^t\cos(2t) + 2e^t\sin(2t) \end{pmatrix} + i\begin{pmatrix} e^t\sin(2t) \\ -2e^t\cos(2t) - e^t\sin(2t) \end{pmatrix}$

$$\boxed{\vec{x} = C_1\begin{pmatrix} e^t\cos(2t) \\ -e^t\cos(2t) + 2e^t\sin(2t) \end{pmatrix} + C_2\begin{pmatrix} e^t\sin(2t) \\ -2e^t\cos(2t) - e^t\sin(2t) \end{pmatrix}}$$

## Problem 6 (40 points)

### Part (a)

Find the Fourier series expansion for the function

$$f(\theta) = \pi^2 - \theta^2, \qquad -\pi < \theta < \pi$$

Write your answer as **both** an infinite sum of complex exponentials and an infinite sum of sines and cosines. [25 points]

(Hint: For $n \neq 0$, $\widehat{f}(n) = -\frac{2(-1)^n}{n^2}$.)

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta$$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi^2 - \theta^2 \, d\theta$$

$$= \frac{1}{2\pi}\left(\pi^2\theta - \frac{\theta^3}{3}\right)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\pi^3 - \frac{\pi^3}{3} - \left(-\pi^3 + \frac{\pi^3}{3}\right)\right)$$

$$= \frac{1}{2\pi}\left(\frac{4\pi^3}{3}\right) = \frac{2\pi^2}{3} \qquad \hat{f}(0) = \frac{2\pi^2}{3}$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi^2 - \theta^2) e^{-in\theta}\, d\theta$$

$u = \theta^2$, $dv = e^{-in\theta} d\theta$
$du = 2\theta\, d\theta$, $v = \frac{e^{-in\theta}}{-in}$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi^2 e^{-in\theta}\, d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta^2 e^{-in\theta}\, d\theta$$

$$= \frac{\pi}{2}\left(\frac{e^{-in\theta}}{in}\right)\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\left(-\frac{\theta^2 e^{-in\theta}}{in} + \frac{2}{in}\int_{-\pi}^{\pi} \theta e^{-in\theta}\, d\theta\right)$$

$u = \theta$, $dv = e^{-in\theta} d\theta$
$du = d\theta$, $v = \frac{e^{-in\theta}}{-in}$

$$= \cancel{\frac{\pi}{2}\left(\frac{e^{-in\theta}}{in}\right)\Big|_{-\pi}^{\pi}} - \frac{1}{2\pi}\left(-\frac{\theta^2 e^{-in\theta}}{in} + \frac{2}{in}\left(-\frac{\theta e^{-in\theta}}{in}\right)\Big|_{-\pi}^{\pi} + \cancel{\frac{2}{in}\frac{1}{in}\int_{-\pi}^{\pi} e^{-in\theta}\, d\theta}\right)$$

$$= -\frac{1}{2\pi}\left(\cancel{-\frac{\pi^2}{in}\left(e^{-in\pi} - e^{+in\pi}\right)}\right) + \frac{2}{in}\left(-\pi\left(\frac{(-1)^n}{in}\right) - \pi\left(\frac{(-1)^n}{in}\right)\right)$$

$$= -\frac{1}{2\pi}\left(+\frac{2\pi}{+n^2}\right)(-1)^n(2) = -\frac{2(-1)^n}{n^2}$$

$$\boxed{f(\theta) \sim \frac{2\pi^2}{3} + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} -\frac{2(-1)^n}{n^2} e^{in\theta}}$$

$$= \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n^2}\left(\cos(n\theta) + \cancel{i\sin(n\theta)}\right)$$

$$+ \sum_{n=1}^{\infty} -\frac{2(-1)^{-n}}{(-n)^2}\left(\cos(-n\theta) + \cancel{i\sin(-n\theta)}\right)$$

$$= \boxed{\frac{2\pi^2}{3} + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2}\cos(n\theta)}$$

**Part (b)**

Use the Fourier series from part (a) to calculate the value of the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

[15 points]

(Hint: Using the hint from the previous part and $\widehat{f}(0)$, it is possible to do this, even if you could not complete the computation in part (a).)

$$\hat{f}(0) = \frac{2\pi^2}{3} \qquad \hat{f}(n) = -\frac{2(-1)^n}{n^2} \text{ for } n \neq 0$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2-\theta^2)^2 d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \pi^4 - 2\pi^2\theta^2 + \theta^4 d\theta$$

$$= \frac{1}{2\pi}\left(\pi^4(2\pi) - 2\pi^2\left(\frac{2\pi^3}{3}\right) + \frac{2\pi^5}{5}\right)$$

$$= \frac{1}{2\pi}\left(2\pi^5 - \frac{4\pi^5}{3} + \frac{2\pi^5}{5}\right)$$

$$= \frac{1}{2\pi}\left(\frac{2\pi^5}{3} + \frac{2\pi^5}{5}\right) = \frac{8\pi^4}{15}$$

$$\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{4\pi^4}{9} + \sum_{\substack{n\neq 0 \\ n\in\mathbb{Z}}} \left|\frac{-2(-1)^n}{n^2}\right|^2$$

$$= \frac{4\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}$$

$$8\sum_{n=1}^{\infty}\frac{1}{n^4} + \frac{4\pi^4}{9} = \frac{8\pi^4}{15}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{8}\left(\frac{8\pi^4}{15} - \frac{4\pi^4}{9}\right)$$

$$= \frac{1}{8}\left(\frac{4\pi^4}{45}\right) = \boxed{\frac{\pi^4}{90}}$$

## Problem 7 (21 points)

Match the following systems of differential equations to the appropriate phase portrait **on the next two pages**. Note that not every figure on the next two pages will be used[1]. [3 points each]

$$x_1'(t) = 3x_1(t) - 5x_2(t)$$
$$x_2'(t) = 5x_1(t) + 3x_2(t) \qquad (1)$$

G

$$x_1'(t) = 3x_1(t) + 2x_2(t)$$
$$x_2'(t) = 2x_1(t) + 3x_2(t) \qquad (2)$$

C

$$x_1'(t) = 2x_2(t)$$
$$x_2'(t) = -2x_1(t) \qquad (3)$$

E

$$x_1'(t) = 3x_1(t)$$
$$x_2'(t) = 3x_2(t) \qquad (4)$$

J

$$x_1'(t) = -3x_1(t) - 4x_2(t)$$
$$x_2'(t) = 4x_1(t) - 3x_2(t) \qquad (5)$$

D

$$x_1'(t) = x_1(t) + x_2(t)$$
$$x_2'(t) = -x_1(t) + 3x_2(t) \qquad (6)$$

A

$$x_1'(t) = 2x_2(t)$$
$$x_2'(t) = 2x_1(t) \qquad (7)$$

F

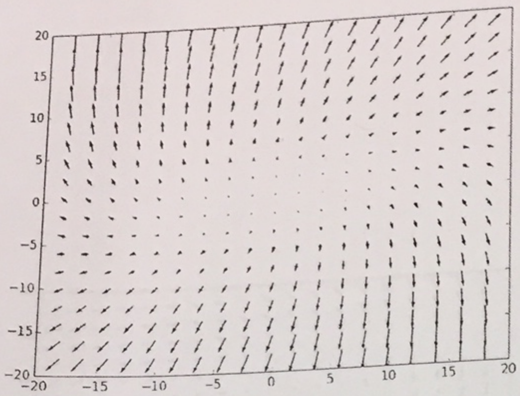
## Bonus (5 points)

On the back of this page, list three facts about properties of harmonic functions (Laplace's equation), Dirichlet eigenfunctions of the Laplacian, the heat equation, or the wave equation from the in-class presentation on partial differential equations.

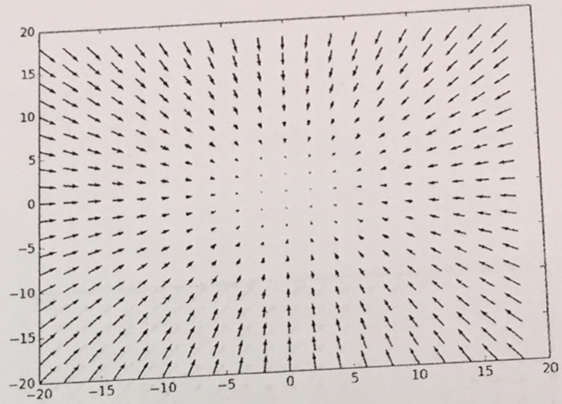
END OF EXAM

(Figures on next two pages)

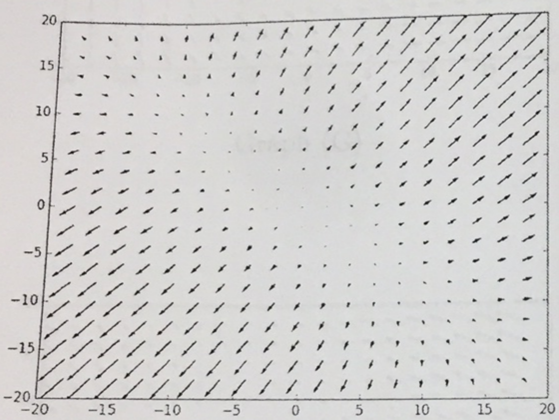
[1] Figures generated using Matplotlib



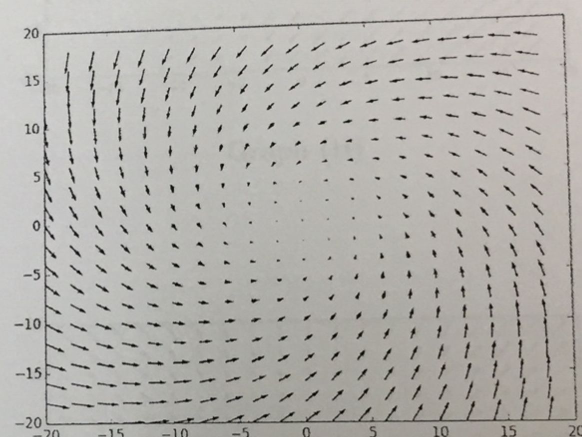

Graph (A)



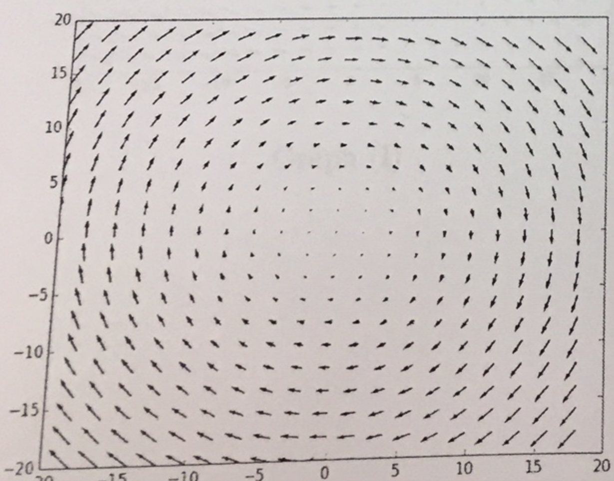

Graph (B)



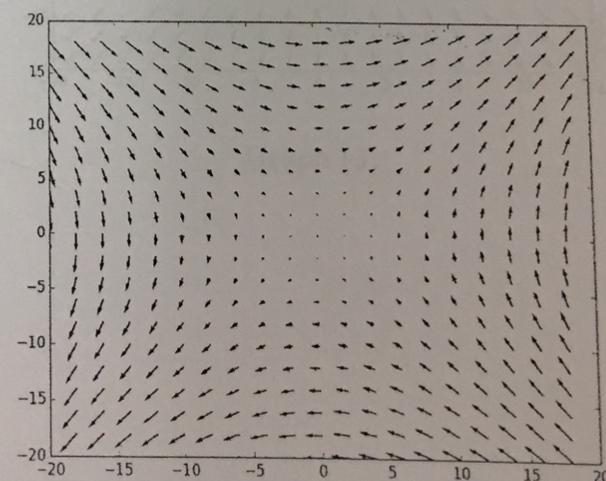

Graph (C)



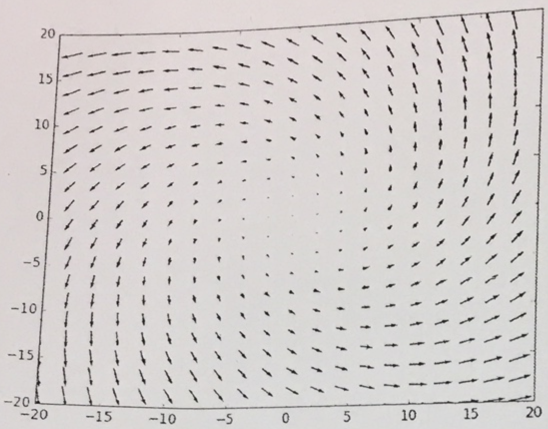

Graph (D)



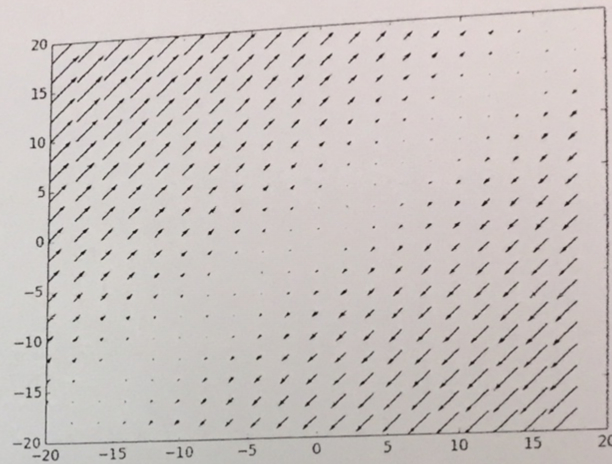

Graph (E)



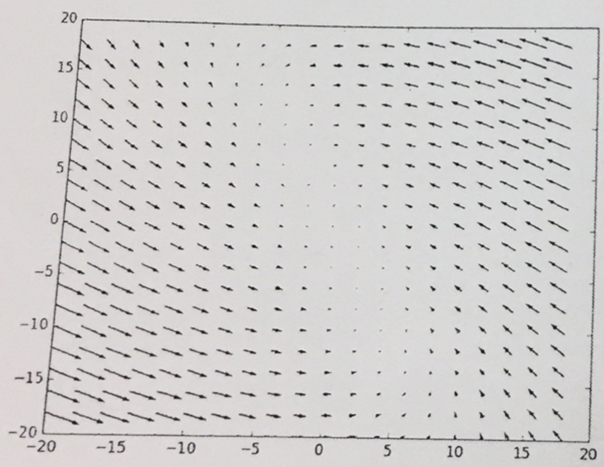

Graph (F)



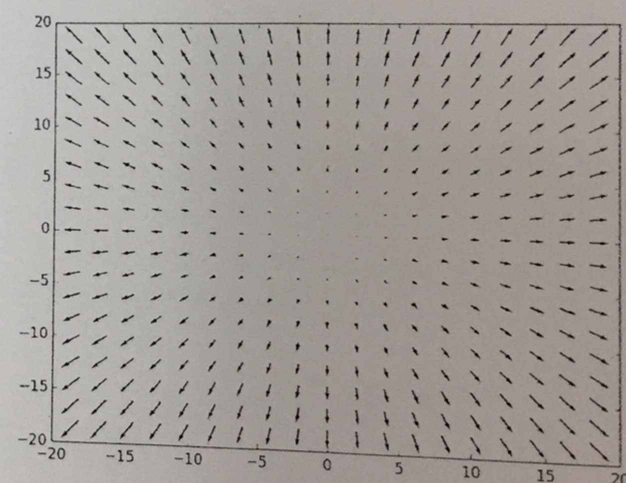

Graph (G)




Graph (H)




Graph (I)




Graph (J)