Math 1B: Discussion 3/5/19 Solutions

Jeffrey Kuan

March 5, 2019

Question 1

First consider

$$\sum_{n=0}^{\infty}(-1)^n = 1 + (-1) + 1 + (-1) + ...$$

This is geometric with $a = 1$ and $r = -1$. Since $|r| \geq 1$, the geometric series diverges.

Next, consider

$$2 + 4 + 6 + 8 + 10 + ... = \sum_{n=1}^{\infty} 2n$$

This series is not geometric, since we are not multiplying by a constant factor to get to the next term we are adding. (Note that it diverges by the nth term test, since $\lim_{n\to\infty} 2n = \infty \neq 0$).

Next, consider

$$\sum_{n=1}^{\infty} \frac{3(-1)^n}{2^n} = \sum_{n=1}^{\infty} 3\left(-\frac{1}{2}\right)^n = -\frac{3}{2} + \frac{3}{4} - \frac{3}{8} + ...$$

This is geometric with $a = -3/2$ and $r = -1/2$. Since $|r| < 1$, the geometric series converges to a value

$$\frac{a}{1-r} = \frac{-3/2}{1-(-1/2)} = -1$$

Next, consider

$$3 - 6 + 12 - 24 + 48 - 96 + ...$$

This is geometric with $a = 3$ and $r = -2$. Since $|r| \geq 1$, this geometric series diverges.

Finally, consider

$$\frac{1}{3} - \frac{1}{6} + \frac{1}{12} - \frac{1}{24} + ...$$

This is geometric with $a = 1/3$ and $r = -1/2$. Since $|r| < 1$, this geometric series converges to a value of

$$\frac{a}{1-r} = \frac{1/3}{1-(-1/2)} = \frac{1/3}{3/2} = \frac{2}{9}$$

Question 2

For the first series,

$$\sum_{n=1}^{\infty} \frac{\text{arctan(n)}}{1+n^2}$$

see the solutions for discussion on 2/28 (this is the same problem).

For the next series,

$$\sum_{n=2}^{\infty} \frac{1}{n^{3/2}\ln(n)}$$

use direct comparison. In particular, $\ln(n) \geq \ln(2)$ for $n \geq 2$. So

$$0 \leq \frac{1}{n^{3/2}\ln(n)} \leq \frac{1}{\ln(2)} \cdot \frac{1}{n^{3/2}}$$

Note that $\sum \frac{1}{\ln(2)} \cdot \frac{1}{n^{3/2}} = \frac{1}{\ln(2)} \sum \frac{1}{n^{3/2}}$ converges by the p-test (since $3/2 > 1$). So the original series converges too by the direct comparison test.

Next, consider

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^{3/2}}$$

The integral test is the way to go here. Consider $f(x) = \frac{1}{x(\ln(x))^{3/2}}$. Note that f is positive for $x \geq 2$ since $\ln(x)$ is positive for $x \geq 2$, and hence so is $(\ln(x))^{3/2}$. It is decreasing since $\ln(x)$ and x are both positive and increasing. (Since $\ln(x)$ is positive and increasing, so is $(\ln(x))^{3/2}$, since $x^{3/2}$ is increasing too and the composition of two increasing functions is increasing). So f is 1 over a positive increasing function and hence is decreasing. f is also continuous since we are considering $x \geq 2$ and the denominator is only zero at $x = 0$ and $x = 1$. So we can use the integral test. Let us first calculate the indefinite integral, using $u = \ln(x)$, $du = \frac{1}{x}dx$,

$$\int \frac{1}{x(\ln(x))^{3/2}}dx = \int \frac{1}{u^{3/2}}du = -2u^{-1/2} + C = -\frac{2}{\sqrt{\ln(x)}} + C$$

So

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln(x))^{3/2}}dx = \text{lim}_{N\to\infty} \int_2^N \frac{1}{x(\ln(x))^{3/2}}dx \\ = \text{lim}_{N\to\infty} - \frac{2}{\sqrt{\ln(N)}} + \frac{2}{\sqrt{\ln(2)}} = 0 + \frac{2}{\sqrt{\ln(2)}} = \frac{2}{\sqrt{\ln(2)}}\end{aligned}$$

where we note that $\sqrt{\ln(N)} \to \infty$ as $N \to \infty$. Since the integral is convergent, so is the original series by the integral test.

Next, consider

$$\sum_{n=1}^{\infty} \frac{e^{-n^2}}{1+3^n}$$

We use direct comparison, noting that e to any negative power is less than or equal to 1.

$$0 \le \frac{e^{-n^2}}{1+3^n} \le \frac{1}{1+3^n} \le \frac{1}{3^n}$$

But note that $\sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ is convergent since it is geometric with $r = 1/3$. So the original series converges by direct comparison.

Next, consider

$$\sum_{n=1}^{\infty} \frac{4+\cos^2(n)}{n^3}$$

Since $-1 \le \cos(n) \le 1$, after squaring, we have that $0 \le \cos^2(n) \le 1$. So

$$4 \le 4 + \cos^2(n) \le 5$$

Therefore,

$$0 \le \frac{4}{n^3} \le \frac{4+\cos^2(n)}{n^3} \le \frac{5}{n^3}$$

Note that $\sum \frac{5}{n^3} = 5 \sum \frac{1}{n^3}$ is convergent by the p-test (since $3 > 1$). So we conclude by direct comparison that the original series converges too.

Next, consider

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2 - n + 4}$$

The "essence" of this series is basically $\sum \frac{\sqrt{n}}{3n^2} = 3 \sum \frac{1}{n^{3/2}}$ so we expect it to converge. There are changes of signs in the denominator, making direct comparison hard. So let us use limit comparison instead, and compare the series to its "essence", which is $\sum \frac{1}{n^{3/2}}$. We have that

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{3n^2-n+4}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2}{3n^2-n+4}$$

$$= \lim_{x\to\infty} \frac{x^2}{3x^2-x+4} = \lim_{x\to\infty} \frac{2x}{6x-1} = \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}$$

where we used L'Hopital's Rule twice on the indeterminate form $\frac{\infty}{\infty}$. Since $c = 1/3 > 0$ is finite, we have that both series converge or diverge. Since $\sum \frac{1}{n^{3/2}}$ converges by the p-test (since $3/2 > 1$), the original series converges too by the limit comparison test.

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

Note that $n! \ge 2^{n-1}$. We can check this for $n = 1$, since $1! = 1 \ge 2^0$. For $n \ge 2$, this is because $n! = n(n-1)(n-2)\cdots(3)(2)$, where we are leaving out the last multiplication by 1. So $n!$ is a product of $n-1$ terms that are all greater than or equal to 2. So $n! \ge 2^{n-1}$. As a concrete example, note that

$$4! = 4 \cdot 3 \cdot 2 \ge 2 \cdot 2 \cdot 2$$

because termwise, we have that $4 \geq 2$, $3 \geq 2$, and $2 \geq 2$. Using the fact that $n! \geq 2^{n-1}$, we have that

$$0 < \frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

But note that $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + 1/2 + 1/4 + 1/8 + ...$ is geometric with $r = 1/2$ and hence converges. So the original series converges too by direct comparison.

Next, consider

$$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

There are two ways to do this. One way is to observe that $n^n \geq n^2$, which works for $n \geq 2$, and can be seen to work for $n = 1$ just by plugging in. So

$$0 < \frac{1}{n^n} \leq \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges, the original series converges too by direct comparison. Another way to do this is to note that for $n \geq 2$,

$$n^n \geq 2^n$$

since n^n is the product of n, n times, where each n is greater than or equal to 2. So n^n is greater than or equal to the product of n twos, by comparing termwise, giving us $n^n \geq 2^n$ whenever $n \geq 2$. For example,

$$4^4 = 4 \cdot 4 \cdot 4 \cdot 4 \geq 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$$

because $4 \geq 2$, comparing the products termwise. So we have that for $n \geq 2$,

$$0 < \frac{1}{n^n} \leq \frac{1}{2^n}$$

But $\sum_{n=2}^{\infty} \frac{1}{2^n} = 1/4 + 1/8 + 1/16 + ...$ converges since it is geometric with $r = 1/2$. So the series $\sum_{n=2}^{\infty} \frac{1}{n^n}$ converges too, and since it does not matter which index we start with, the original series converges too.

Next, consider

$$\sum_{n=4}^{\infty} \frac{\sec^2\left(\frac{\pi}{n}\right)}{n^2}$$

Note that for $n \geq 4$, we have that

$$0 < \frac{\pi}{n} \leq \frac{\pi}{4}$$

Looking at a graph of secant, we have that for $n \geq 4$,

$$1 \leq \sec(\pi/n) \leq \sqrt{2}$$

where $0 = \sec(0)$ and $\sqrt{2} = \sec(\pi/4)$, because $\sec(x)$ is increasing on $[0, \pi/2)$. So in particular, for $n \geq 4$,

$$1 \leq \sec^2(\pi/n) \leq 2$$

by squaring the above inequality. So note that for $n \geq 4$,

$$0 \leq \frac{1}{n^2} \leq \frac{\sec^2\left(\frac{\pi}{n}\right)}{n^2} \leq \frac{2}{n^2}$$

But since $\sum \frac{2}{n^2} = 2\sum \frac{1}{n^2}$ converges by the p-test, the original series converges too by direct comparison.

Finally, consider

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{2\sqrt{n}-1}$$

Note that $\arctan(x)$ is increasing up to a horizontal asymptote at $\pi/2$. So for $n \geq 1$, we have that

$$\frac{\pi}{4} = \arctan(1) \leq \arctan(n) \leq \frac{\pi}{2}$$

The essence of this series is basically a bounded number divided by $2\sqrt{n}$, so it behaves like $\frac{1}{2}\sum \frac{1}{\sqrt{n}}$ times some constant. This series is divergent, so we guess that the original series is divergent. To prove this, use the above inequality.

$$\frac{\pi}{4} \leq \arctan(n) \leq \frac{\pi}{2}$$

Since we want to show divergence, we want to cut in from below. So note that

$$\frac{\arctan(n)}{2\sqrt{n}-1} \geq \frac{\arctan(n)}{2\sqrt{n}} \geq \frac{\pi}{4} \cdot \frac{1}{2\sqrt{n}} = \frac{\pi}{8}\frac{1}{\sqrt{n}} > 0$$

Note that $\sum \frac{\pi}{8}\frac{1}{\sqrt{n}} = \frac{\pi}{8}\sum \frac{1}{\sqrt{n}}$ diverges by the p-test (since $1/2 \leq 1$). So the original series diverges by direct comparison.