# Math 1B: Discussion 2/28/19 Solutions <br> Jeffrey Kuan 

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## Question 1

We consider the series

$$
\sum_{n=2}^{\infty} \frac{1}{n^{2}-n}
$$

Let's begin by writing out a few terms of the infinite sum.

$$
\sum_{n=2}^{\infty} \frac{1}{n^{2}-n}=\frac{1}{2^{2}-2}+\frac{1}{3^{2}-3}+\frac{1}{4^{2}-4}+\ldots=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\ldots
$$

The first few partial sums are

$$
\begin{gathered}
S_{1}=\frac{1}{2} \\
S_{2}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3} \\
S_{3}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}
\end{gathered}
$$

Let's do a partial fraction decomposition for $\frac{1}{x^{2}-x}=\frac{1}{x(x-1)}$.

$$
\begin{gathered}
\frac{1}{x^{2}-x}=\frac{A}{x}+\frac{B}{x-1} \\
1=A(x-1)+B x \\
0 x+1=(A+B) x-A
\end{gathered}
$$

So matching coefficients, we get

$$
\begin{gathered}
A+B=0 \\
-A=1
\end{gathered}
$$

So $A=-1, B=1$. Thus,

$$
\frac{1}{x^{2}-x}=\frac{1}{x-1}-\frac{1}{x}
$$

Using this partial fraction decomposition, we have that

$$
\sum_{n=2}^{\infty} \frac{1}{n^{2}-n}=\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)
$$

Writing out a few partial sums (same as before but written differently in a more convenient way now),

$$
\begin{gathered}
S_{1}=\left(1-\frac{1}{2}\right)=1-\frac{1}{2} \\
S_{2}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)=1-\frac{1}{3} \\
S_{3}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)=1-\frac{1}{4}
\end{gathered}
$$

The cancellations here explain why this is called a telescoping series. From the previous calculations, we can see the following pattern.

$$
S_{k}=1-\frac{1}{k+1}
$$

To see whether the series diverges or converges, just consider the limit of the partial sums $S_{k}$.

$$
\lim _{k \rightarrow \infty} S_{k}=\lim _{k \rightarrow \infty}\left(1-\frac{1}{k+1}\right)=1
$$

Since this limit exists, we have that the original series converges and has value 1.

## Question 2

- First consider

$$
\sum_{n=1}^{\infty} \frac{1}{4+n^{2}}
$$

Let's first try the integral test. Indeed, we have that $f(x)=\frac{1}{4+x^{2}}$ is nonnegative and continuous (since its denominator is never zero). It is also decreasing since the denominator is a positive increasing function. So we can use the integral test.

$$
\int \frac{1}{4+x^{2}} d x=\frac{1}{4} \int \frac{1}{1+(x / 2)^{2}} d x=\frac{1}{2} \arctan \left(\frac{x}{2}\right)+C
$$

where the last integral can be done using $u=x / 2$. We then calculate

$$
\begin{aligned}
\int_{1}^{\infty} \frac{1}{4+x^{2}} d x & =\left.\lim _{N \rightarrow \infty} \frac{1}{2} \arctan \left(\frac{N}{2}\right)\right|_{1} ^{N} \\
& =\lim _{N \rightarrow \infty}\left(\frac{1}{2} \arctan (N / 2)-\frac{1}{2} \arctan (1 / 2)\right)=\frac{1}{2} \cdot \frac{\pi}{2}-\frac{1}{2} \arctan (1 / 2)
\end{aligned}
$$

which is a finite value. So the integral converges and hence the original series converges.
This can also be done with Direct Comparison. Note that $4+n^{2}>n^{2}$. So

$$
0<\frac{1}{4+n^{2}}<\frac{1}{n^{2}}
$$

But since $\sum_{n=1}^{\infty} \frac{1}{n^{2}}$ converges by the $p$-test (since the exponent on $n$ is larger than 1 ), the original series $\sum_{n=1}^{\infty} \frac{1}{4+n^{2}}$ also converges by Direct Comparison.

- Next, consider

$$
\sum_{n=1}^{\infty} \frac{\arctan (n)}{1+n^{2}}
$$

Let's first try the integral test. Indeed, the function $f(x)=\frac{\arctan (x)}{1+x^{2}}$ is positive and continuous. To check it is decreasing, compute

$$
f^{\prime}(x)=\frac{1-2 x \arctan (x)}{\left(1+x^{2}\right)^{2}}
$$

One can see that $f^{\prime}(x)$ is always negative for $x \geq 1(\operatorname{since} \arctan (x)$ and $2 x$ are both increasing, and one can compute $\left.f^{\prime}(1)<0\right)$. So $f$ is decreasing for $x \geq 1$, as desired. So we can use the integral test.

Using a $u$-substitution, we find that

$$
\int \frac{\arctan (x)}{1+x^{2}} d x=\frac{1}{2}(\arctan (x))^{2}+C
$$

So we have that

$$
\begin{aligned}
\int_{1}^{\infty} \frac{\arctan (x)}{1+x^{2}} d x= & \left.\lim _{N \rightarrow \infty} \frac{1}{2}(\arctan (x))^{2}\right|_{1} ^{N} \\
& =\lim _{N \rightarrow \infty} \frac{1}{2}(\arctan (N))^{2}-\frac{1}{2}(\arctan (1))^{2}=\frac{1}{2}\left(\frac{\pi}{2}\right)^{2}-\frac{1}{2}\left(\frac{\pi}{4}\right)^{2}
\end{aligned}
$$

So the integral converges. Hence, the original series converges by the integral test.
We can also use Direct Comparison, since the terms of the series are nonnegative. Note that $0<\arctan (n)<\frac{\pi}{2}$ for $n \geq 1$ (by the horizontal asymptote of $\arctan$ ). So

$$
0<\frac{\arctan (n)}{1+n^{2}}<\frac{\pi}{2} \cdot \frac{1}{1+n^{2}}<\frac{\pi}{2} \frac{1}{n^{2}}
$$

But note that

$$
\sum_{n=1}^{\infty} \frac{\pi}{2} \frac{1}{n^{2}}=\frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^{2}}
$$

converges by the $p$ test (since the exponent on $n$ is 2 , which is bigger tan 1 ). So the original series converges by Direct Comparison.

- Next, consider

$$
\sum_{n=2}^{\infty} \frac{1}{n^{2}-1}
$$

Let's first try the integral test. One can check that the function $f(x)=\frac{1}{x^{2}-1}$ is nonnegative, continuous, and decreasing. Using a partial fraction decomposition,

$$
\int \frac{1}{x^{2}-1} d x=\int \frac{1}{2} \cdot \frac{1}{x-1}-\frac{1}{2} \cdot \frac{1}{x+1} d x=\frac{1}{2} \ln |x-1|-\frac{1}{2} \ln |x+1|+C
$$

Then, note that

$$
\int_{2}^{\infty} \frac{1}{x^{2}-1} d x=\lim _{N \rightarrow \infty}\left(\frac{1}{2} \ln |N-1|-\frac{1}{2} \ln |N+1|-\frac{1}{2} \ln (1)+\frac{1}{2} \ln (3)\right)
$$

We plug in and get an $\infty-\infty$. This is indeterminate so we have to do something to fix this. But we can just use properties of logarithms to get that the limit above is the same as

$$
\begin{aligned}
\int_{2}^{\infty} \frac{1}{x^{2}-1} d x=\lim _{N \rightarrow \infty} & \left(\frac{1}{2} \ln |N-1|-\frac{1}{2} \ln |N+1|-\frac{1}{2} \ln (1)+\frac{1}{2} \ln (3)\right) \\
& =\lim _{N \rightarrow \infty}\left(\frac{1}{2} \ln \left|\frac{N-1}{N+1}\right|-\frac{1}{2} \ln (1)+\frac{1}{2} \ln (3)\right)=\frac{1}{2} \ln (3)
\end{aligned}
$$

where we used the fact that $\frac{N-1}{N+1} \rightarrow 1$ as $N \rightarrow \infty$, and $\ln (1)=0$. So the integral converges and hence the original series converges.

It is hard to use Direct Comparison since

$$
\frac{1}{n^{2}-1}>\frac{1}{n^{2}}>0
$$

so the comparison is going the wrong way, since $\sum \frac{1}{n^{2}}$ converges by the $p$-test (since $2>1$ ). On Tuesday, we will learn the Limit Comparison Test, which will handle this case well.

- Next, consider

$$
\sum_{n=0}^{\infty} \frac{n^{3}}{n^{4}-5}
$$

First, try the integral test. Note that $f(x)=\frac{x^{3}}{x^{4}-5}$ is not positive or continuous for $x \geq 0$ (the denominator has a zero at the fourth root of 5 ). But it is eventually positive and continuous for $x \geq 2$. So rewrite

$$
\sum_{n=0}^{\infty} \frac{n^{3}}{n^{4}-5}=0-\frac{1}{4}+\sum_{n=2}^{\infty} \frac{n^{3}}{n^{4}-5}
$$

and note that the convergence or divergence of the original sequence is equivalent to the convergence or divergence of the shifted sequence $\sum_{n=2}^{\infty} \frac{n^{3}}{n^{4}-5}$. Finally, we just need to check whether $f$ is decreasing for $x \geq 2$.

$$
f^{\prime}(x)=\frac{3 x^{2}\left(x^{4}-5\right)-4 x^{6}}{\left(x^{4}-5\right)^{2}}=\frac{-x^{6}-16 x^{2}}{\left(x^{4}-5\right)^{2}}
$$

The numerator is always negative (since even powers are always positive) and the denominator is always positive (since it is a square of something). So $f^{\prime}$ is always negative. So $f$ is decreasing.

So we can use the integral test. We can calculate using a $u$-substitution, $u=x^{4}-5$, that

$$
\int \frac{x^{3}}{x^{4}-5} d x=\frac{1}{3} \ln \left|x^{4}-5\right|+C
$$

So

$$
\int_{2}^{\infty} \frac{x^{3}}{x^{4}-5} d x=\lim _{N \rightarrow \infty} \frac{1}{3} \ln \left|N^{4}-5\right|-\frac{1}{3} \ln (11)=\infty
$$

since $N^{4}-5 \rightarrow \infty$ as $N \rightarrow \infty$ and $\ln |x| \rightarrow \infty$ as $x \rightarrow \infty$. The integral diverges, so the original series diverges.

We can also use Direct Comparison on the series $\sum_{n=2}^{\infty} \frac{n^{3}}{n^{4}-5}$, where we write

$$
\sum_{n=0}^{\infty} \frac{n^{3}}{n^{4}-5}=0-\frac{1}{4}+\sum_{n=2}^{\infty} \frac{n^{3}}{n^{4}-5}
$$

and note that the terms $\frac{n^{3}}{n^{4}-5}$ are indeed nonnegative for $n \geq 2$. We have that

$$
\frac{n^{3}}{n^{4}-5}>\frac{n^{3}}{n^{4}}=\frac{1}{n}>0
$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the $p$ test (since the exponent on $n$, which is 1 , is less than or equal to 1), the original series diverges by Direct Comparison.

- Next, consider

$$
\sum_{n=1}^{\infty} \frac{4+\sin \left(n^{2}\right)}{n \sqrt{n}}
$$

It is hard to integrate $f(x)$ here, so let's not do the integral test. Let's try Direct Comparison.

This series is "basically" like a constant times $\sum_{n=1}^{\infty} \frac{1}{n \sqrt{n}}=\sum_{n=1}^{\infty} \frac{1}{n^{3 / 2}}$. So we would expect this to converge by the $p$ test (since $3 / 2>1$ ). So let's try to show using Direct Comparison that the original series converges. Note that

$$
-1 \leq \sin \left(n^{2}\right) \leq 1
$$

$$
3 \leq 4+\sin \left(n^{2}\right) \leq 5
$$

So in particular,

$$
0<\frac{3}{n \sqrt{n}} \leq \frac{4+\sin \left(n^{2}\right)}{n \sqrt{n}} \leq \frac{5}{n \sqrt{n}}
$$

(where the bottom inequality shows that the terms of our series are nonnegative so we can use Direct Comparison). Note that $\sum_{n=1}^{\infty} \frac{5}{n \sqrt{n}}=5 \sum_{n=1}^{\infty} \frac{1}{n^{3 / 2}}$ converges by the $p$-test, so the original series converges by Direct Comparison.

- Finally, consider

$$
\sum_{n=1}^{\infty} \frac{1}{n!}
$$

We observe that $n!\geq 2^{n-1}$. To see this, we can write out

$$
\begin{gathered}
2!=2 \geq 2 \\
3!=3 \cdot 2 \geq 2 \cdot 2=2^{2} \\
4!=4 \cdot 3 \cdot 2 \geq 2 \cdot 2 \cdot 2=2^{3}
\end{gathered}
$$

(since $4 \geq 2,3 \geq 2$, and $2 \geq 2$, so $4 \cdot 3 \cdot 2 \geq 2 \cdot 2 \cdot 2$ ). So in particular,

$$
n!=n \cdot(n-1) \cdot(n-2) \cdot \ldots \cdot 3 \cdot 2
$$

where when we are leaving out the multiplication by 1 , we are multiplying $n-1$ numbers that are all $\geq 2$. So $n!\geq 2^{n-1}$.

We cannot integrate a factorial function. But we can use Direct Comparison. Note that by the inequality we just showed,

$$
0<\frac{1}{n!}<\frac{1}{2^{n-1}}=2^{1-n}
$$

So we just need to show that

$$
\sum_{n=1}^{\infty} 2^{1-n}
$$

converges. But since $f(x)=2^{1-x}$ is positive, continuous, and decreasing (look at at a graph of $f(x)$ ), we can use the integral test. Remembering that $\left(2^{x}\right)^{\prime}=\ln (2) \cdot 2^{x}$, we have that

$$
\int 2^{1-x} d x=-\int 2^{u} d u=-\frac{1}{\ln (2)} 2^{u}+C=-\frac{1}{\ln (2)} 2^{1-x}+C
$$

where we used $u=1-x$. So we have that

$$
\int_{1}^{\infty} 2^{1-x} d x=\lim _{N \rightarrow \infty}\left(-\frac{1}{\ln (2)} 2^{1-N}+\frac{1}{\ln (2)}\right)=\frac{1}{\ln (2)}
$$

since $2^{1-N} \rightarrow 0$ as $N \rightarrow \infty$, since we are taking 2 to a very large negative power. So we have that

$$
\sum_{n=1}^{\infty} 2^{1-n}
$$

converges by the integral test. So by Direct Comparison, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges too.

## Question 3

We have that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the $p$ test. Or we can use the integral test. Note that $f(x)=1 / x$ is positive, continuous, and decreasing for $x \geq 1$. Then,

$$
\int_{1}^{\infty} \frac{1}{x} d x=\lim _{N \rightarrow \infty} \ln |N|-\ln |1|=\infty
$$

The integral diverges, so the original series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges too.
Next, consider $\sum_{n=2}^{\infty} \frac{1}{n \ln (n)}$. We note that

$$
f(x)=\frac{1}{x \ln (x)}
$$

is positive (since $\ln (x)>0$ for $x \geq 2$ ), continuous, and decreasing (since $x$ and $\ln (x)$ are positive increasing functions on $x \geq 2$ ), on $x \geq 2$. So we can use the integral test. We can calculate using a $u$-substitution, $u=\ln (x)$, that

$$
\int \frac{1}{x \ln (x)} d x=\ln |\ln (x)|+C
$$

So we have that

$$
\int_{2}^{\infty} \frac{1}{x \ln (x)} d x=\lim _{N \rightarrow \infty} \ln |\ln (N)|-\ln |\ln (2)|=\infty
$$

since $\ln (N) \rightarrow \infty$ as $N \rightarrow \infty$, so $\ln |\ln (N)| \rightarrow \infty$ also as $N \rightarrow \infty$. So since the integral diverges, we have that

$$
\sum_{n=2}^{\infty} \frac{1}{n \ln (n)}
$$

diverges too by the integral test.
Finally, consider

$$
\sum_{n=2}^{\infty} \frac{1}{n(\ln (n))^{2}}
$$

Let $f(x)=\frac{1}{x(\ln (x))^{2}}$. For $x \geq 2$, this is positive, continuous, and decreasing (since $x$ and $\ln (x)$ and hence $(\ln (x))^{2}$ are positive increasing functions for $x \geq 2$ ). So we can use the integral test. Use a $u$-substitution $u=\ln (x)$ to calculate

$$
\int \frac{1}{x(\ln (x))^{2}} d x=\int \frac{1}{u^{2}} d u=-\frac{1}{u}+C=-\frac{1}{\ln (x)}+C
$$

Then, we calculate

$$
\int_{2}^{\infty} \frac{1}{x(\ln (x))^{2}} d x=\lim _{N \rightarrow \infty}\left(-\frac{1}{\ln (N)}+\frac{1}{\ln (2)}\right)=0+\frac{1}{\ln (2)}=\frac{1}{\ln (2)}
$$

where $\frac{1}{\ln (N)}$ as $N \rightarrow \infty$ is 0 since it is 1 over a very large number. So the integral converges and hence the original series $\sum_{n=2}^{\infty} \frac{1}{n(\ln (n))^{2}}$ converges. So the extra natural $\log$ in the denominator really does make a difference!

