Math 1B: Discussion 2/26/19 Answers

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February 26, 2019

Question 1

First consider

$$a_n = \frac{1}{n^{1/8}}$$

We have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{1/8}} = 0$$

Next, consider

$$a_n = \frac{(-1)^{n^2}}{\sqrt{n}}$$

Recall from class that $\lim_{n\to\infty} |a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$. Here, we have that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^{n^2}}{\sqrt{n}} \right| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

So since $\lim_{n\to\infty} |a_n| = 0$, we have that $\lim_{n\to\infty} a_n = 0$.

For the next sequence

$$a_n = (-1)^n 2^n$$

we see that the sequence is $-2, 4, -8, 16, ...$ so this sequence cannot converge.

Next, consider

$$a_n = (-1)^n 2^{-n}$$

We see that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |(-1)^n 2^{-n}| = \lim_{n\to\infty} 2^{-n} = 0$$

since 2^{-n} is $\left(\frac{1}{2}\right)^n$.

Next, consider

$$a_n = \cos\left(\frac{1}{n^2}\right)$$

Note that $1/n^2 \to 0$ as $n \to \infty$. So since $\cos(x)$ is a continuous function, we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos\left(\lim_{n\to\infty} \frac{1}{n^2}\right) = \cos(0) = 1$$

Finally, consider

$$a_n = \frac{\sin^3(n^2)}{\sqrt{n}}$$

Let us show that $\lim_{n\to\infty}|a_n| = 0$. Use the squeeze theorem.

$$0 \leq |\sin(n^2)| \leq 1$$

so

$$0 \leq |\sin^3(n^2)| \leq 1$$

$$0 \leq \frac{|\sin^3(n^2)|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

But note that

$$\lim_{n\to\infty} 0 = 0$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

So by the squeeze theorem,

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} \frac{\left|sin^3(n^2)\right|}{\sqrt{n}} = 0$$

Thus, since $\lim_{n\to\infty}|a_n| = 0$, we also have that $\lim_{n\to\infty} a_n = 0$.

Question 2

We have that

$$S_1 = -\frac{1}{3}$$

$$S_2 = -\frac{1}{3} + \frac{1}{9}$$

$$S_3 = -\frac{1}{3} + \frac{1}{9} - \frac{1}{27}$$

and so on. Remember the partial sum formula for a geometric series.

$$a + ar + ar^2 + ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

Note that

$$S_k = -\frac{1}{3} + \frac{1}{9} - ... + \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{k-1}$$

Using the partial sum formula (where $a = -1/3$ since a is the first term, $r = -1/3$ since r is the multiplication factor), we have that

$$S_k = \frac{-1/3(1-(-1/3)^k)}{(1-(-1/3)} = -\frac{1}{3}\cdot\frac{3}{4}\left(1-\left(-\frac{1}{3}\right)^k\right) = -\frac{1}{4}\left(1-\left(-\frac{1}{3}\right)^k\right)$$

Then, to find out whether this series converges or diverges, we compute $\lim_{k\to\infty} S_k$.

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} -\frac{1}{4}\left(1-\left(-\frac{1}{3}\right)^k\right) = -\frac{1}{4}(1-0) = -\frac{1}{4}$$

since $(-1/3)^k \to 0$ as $k \to \infty$, since multiply $-1/3$ to itself over and over again makes it smaller in magnitude (ignoring the sign). So the geometric series converges (which we could also immediately see from the fact that $|r| < 1$ since $r = -1/3$).

Question 3

Consider the first series

$$\sum_{n=1}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + ...$$

You can tell it diverges in one of the following three ways.

- It is a geometric series with $r = -1$. Since $|r| \geq 1$, this geometric series diverges.
- Let $a_n = (-1)^n$. We see that $\lim_{n\to\infty} a_n$ does not exist since it bounces back and forth between 1 and -1. So since $\lim_{n\to\infty} a_n \neq 0$, this series diverges by the nth term test.
- If you calculate the partial sums, you get

$$S_1 = 1$$

$$S_2 = 1 + (-1) = 0$$
$$S_3 = 1 + (-1) + 1 = 1$$
$$S_4 = 1 + (-1) + 1 + (-1) = 0$$

  So the partial sums bounce back and forth between 1 and 0, and hence $\lim_{k\to\infty} S_k$ does not exist. So the series diverges.

Next, consider

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n = \sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = -\frac{2}{3} + \frac{4}{9} - \frac{8}{27} + ...$$

Note that this series is geometric with $r = -2/3$. So since $|r| < 1$, this geometric series converges. The value it converges to is given by the formula

$$\frac{a}{1-r} = \frac{-2/3}{1-(-2/3)} = -\frac{2}{5}$$

Next, consider

$$\sum_{n=1}^{\infty}(-1)^{n^2}\frac{2n^2}{n^2+4}$$

We show that this diverges by the nth term test. In particular, for

$$a_n = (-1)^{n^2}\frac{2n^2}{n^2+4}$$

we claim that $\lim_{n\to\infty} a_n = 0$. Remember that $\lim_{n\to\infty}|a_n| = 0$ if and only if $\lim_{n\to\infty} a_n = 0$. So $\lim_{n\to\infty}|a_n| \neq 0$ if and only if $\lim_{n\to\infty} a_n \neq 0$. We can easily see that

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{2n^2}{n^2+4} = 2$$

(can be shown by L'Hopital, or by dividing top and bottom by n^2). So since $\lim_{n\to\infty}|a_n| = 2 \neq 0$, we conclude that $\lim_{n\to\infty} a_n \neq 0$ also. So by the nth term test, this series diverges.

Finally, consider

$$\sum_{n=1}^{\infty}\frac{1}{3^n+2}$$

Note that $3^n + 2 > 3^n$, so

$$0 < \frac{1}{3^n+2} < \frac{1}{3^n}$$

The series $\sum_{n=1}^{\infty}\frac{1}{3^n} = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is geometric with $r = 1/3$ and hence converges since $|r| < 1$. Then by comparison,

$$\sum_{n=1}^{\infty}\frac{1}{3^n+2}$$

converges also. It is not possible to find its actual value, at least using the methods we have so far.

Question 4

Suppose we know that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Then, note that for positive integers n,

$$n \geq \sqrt{n}$$

So,

$$\frac{1}{n} \leq \frac{1}{\sqrt{n}}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n} \leq \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

But since we know that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges to infinity, we have by the above inequality that $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges to infinity also. (This is the comparison test).

Now, suppose we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to some value M. Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

We could show that this converges by the comparison test. But let's use the following more hands-on approach. Let S_k denote the partial sums of

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

Note that

$$\frac{1}{n^3} \leq \frac{1}{n^2}$$

for all positive integers n. So we have that the partial sums S_k must be bounded above by $M = \sum_{n=1}^{\infty} \frac{1}{n^2}$. In addition, S_k is monotonically increasing since all terms that are being added in the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$ are positive. So S_k (the sequence of partial sums for $\sum_{n=1}^{\infty} \frac{1}{n^3}$) is a monotonically increasing sequence that is bounded above by M. So it must converge. Since $\lim_{k\to\infty} S_k$ thus exists, we have by definition that the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges.