KEY

Math 1A: Derive That Derivative!

Jeffrey Kuan

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Team Name: The Answer Key

Team Member Names: Jeffrey Kuan

Welcome to America's favorite derivative-based game show, Derive That Derivative! You and your team of four people will have 40 minutes to tackle 20 of the gnarliest derivatives out there. Here are the rules.

- In each question, you will calculate the derivative of the given function, for the number of points specified (4 points, 6 points, 8 points, or 10 points).
- You will be given a score based on the following rubric.
  - Full (100%) credit: Derivative is calculated correctly, or almost completely correctly.
  - Almost complete (75%) credit: Derivative is calculated with a few minor mistakes.
  - Half (50%) credit: Derivative is calculated with several mistakes.
  - Almost no (25%) credit: Derivative is calculated with major mistakes.
  - No (0%) credit: No significant attempt.
- Each question also has a bonus question that is related to the given function.
  - If you answer the bonus question correctly, you get 1.5 times as many points for that question, but ONLY if you got full credit for that derivative.
  - So if Question X is worth 4 points, if you answer Question X and its bonus correctly, you will get a total of 6 points.
  - But if you answer the derivative for Question X incorrectly, even if you get the bonus question for Question X right, you will get no points for the bonus. So it is important to try to do the derivatives correctly, so that you can get the bonus points too.
- The maximum possible score is 180 points.
- Good luck and have fun!

Question 1 (4 points)

Find the derivative of
$$f(x) = 3x + \frac{2}{\sqrt[3]{x}} + \frac{1}{\sqrt{x}} + e^{-x}$$

$$f(x) = 3x + 2x^{-\frac{1}{3}} + x^{-\frac{1}{2}} + e^{-x}$$

$$f'(x) = 3 - \frac{2}{3}x^{-\frac{4}{3}} - \frac{1}{2}x^{-\frac{3}{2}} - e^{-x}$$

Bonus: What is $\lim_{x\to\infty} f'(x)$?

$$\lim_{x\to\infty}\left(3 - \underbrace{\frac{2}{3}x^{-\frac{4}{3}}}_{\to 0} - \underbrace{\frac{1}{2}x^{-\frac{3}{2}}}_{\to 0} - \underbrace{e^{-x}}_{\to 0}\right)$$

$$= \boxed{3}$$

Question 2 (4 points)

Find the derivative of
$$f(x) = \arctan(x^2 + 1)$$

$$f'(x) = \frac{1}{1+(x^2+1)^2} \cdot 2x$$

$$= \frac{2x}{1+(x^2+1)^2}$$

Bonus: What is the domain of $f(x)$?

Domain of arctan(x) is $\mathbb{R}$, so $\boxed{\mathbb{R}}$.

Question 3 (4 points)

Find the derivative of

$$f(x) = e^{(e^{-x})} + e^{(-e^{x})}$$

$$f'(x) = -e^{(e^{-x})} \cdot e^{-x} + e^{(-e^{x})} \cdot (-e^{x})$$

Bonus: What is $\lim_{x\to\infty} f(x)$?

$$\lim_{x\to\infty} \left(e^{(e^{-x})} + e^{(-e^{x})}\right)$$

$$= e^{0} + e^{\overbrace{\lim_{x\to\infty} -e^{x}}^{\to -\infty}} = 1 + 0 = \boxed{1}$$

Question 4 (4 points)

Find the derivative of

$$f(x) = 2^{3x} + 3^{2x} + 4^{x}$$

$$f(x) = 8^{x} + 9^{x} + 4^{x}$$

$$f'(x) = \ln 8 \cdot 8^{x} + \ln 9 \cdot 9^{x} + \ln 4 \cdot 4^{x}.$$

Bonus: $f'(1)$ can be written as $\ln(C)$ for some positive integer C. What is C?

$$f'(1) = 8 \ln 8 + 9 \ln 9 + 4 \ln 4$$

$$= \ln(8^{8} \cdot 9^{9} \cdot 4^{4})$$

$$\boxed{C = 4^{4} \cdot 8^{8} \cdot 9^{9}}$$

Question 5 (4 points)

Find the derivative of

$$f(x) = \sqrt{e^x + e^{3x} + e^{5x}}$$

$$f'(x) = \frac{e^x + 3e^{3x} + 5e^{5x}}{2\sqrt{e^x + e^{3x} + e^{5x}}}$$

Bonus: What is the equation of the tangent line to $f(x)$ at $x = 0$?

$$f(0) = \sqrt{3}, \quad f'(0) = \frac{9}{2\sqrt{3}}$$

$$\boxed{y - \sqrt{3} = \frac{9}{2\sqrt{3}}(x - 0)}$$

Question 6 (4 points)

Find the derivative of

$$f(x) = \ln(\sqrt{x}) + \ln(x^2) + \ln(x^4) + \ln(x^8)$$

$$f(x) = \frac{1}{2}\ln(x) + 2\ln(x) + 4\ln(x) + 8\ln(x)$$

$$= \frac{29}{2}\ln(x)$$

$$f'(x) = \frac{29}{2x}.$$

Bonus: At what value of a does the normal line to $f(x)$ at $x = a$ have slope $-1/29$?

normal line has slope $-\frac{1}{29}$,

tangent line has slope 29.

$$\frac{29}{2x} = 1 \qquad x = \frac{1}{2} \qquad \boxed{a = \frac{1}{2}}$$

Question 7 (4 points)

Find the derivative of

$$f(x) = \arctan\left(\frac{3}{\pi}\arcsin(x)\right)$$

$$f'(x) = \frac{1}{1+\left(\frac{3}{\pi}\arcsin x\right)^2} \cdot \frac{3}{\pi} \cdot \frac{1}{\sqrt{1-x^2}}.$$

Bonus: What is $f\left(\frac{1}{2}\right)$?

$$f\left(\frac{1}{2}\right) = \arctan\left(\frac{3}{\pi}\arcsin\left(\frac{1}{2}\right)\right)$$

$$= \arctan\left(\frac{3}{\pi} \cdot \frac{\pi}{6}\right) = \boxed{\arctan\left(\frac{1}{2}\right)}$$

Question 8 (4 points)

Find the derivative of

$$f(x) = \ln(\ln(\ln(x)))$$

$$f'(x) = \frac{1}{\ln(\ln(x))} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

Bonus: Find a value of a such that $f(a) = 4$.

$$a = e^{(e^{(e^4)})}$$

Question 9 (6 points)

Find the derivative of

$$f(x) = \frac{\arcsin(e^{-2x})}{x^2 - 1}$$

$$f'(x) = \frac{(x^2-1)\frac{1}{\sqrt{1-e^{-4x}}}\cdot(-2e^{-2x}) - 2x\arcsin(e^{-2x})}{(x^2-1)^2}$$

Bonus: What is the domain of $f(x)$?

$x^2 - 1 \neq 0, \quad x \neq -1, 1$

$-1 \leq e^{-2x} \leq 1 \longrightarrow x \geq 0$

So domain is $\boxed{[0,1)\cup(1,\infty)}$.

Question 10 (6 points)

Find the derivative of

$$f(x) = \frac{\frac{1}{x^2}}{1 + \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6}}$$

For this question, simplify your answer (no fractions within fractions, simplify the numerator). You can leave the denominator of your derivative as the square of a quantity.

$$f(x) = \frac{x^4}{1+x^2+x^4+x^6}$$

$$f'(x) = \frac{4x^3(1+x^2+x^4+x^6) - x^4(2x+4x^3+6x^5)}{(1+x^2+x^4+x^6)^2}$$

$$= \frac{4x^3 + 4x^5 \cancel{+4x^7} + 4x^9 - 2x^5 \cancel{-4x^7} - 6x^9}{(1+x^2+x^4+x^6)^2}$$

$$= \frac{4x^3 + 2x^5 - 2x^9}{(1+x^2+x^4+x^6)^2}$$

Bonus: What is $\lim_{x\to\infty} f(x)$?

$$\lim_{x\to\infty} \frac{x^4}{1+x^2+x^4+x^6}$$

$$= \lim_{x\to\infty} \frac{\frac{x^4}{x^6}}{\frac{1}{x^6}+\frac{x^2}{x^6}+\frac{x^4}{x^6}+\frac{x^6}{x^6}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{\frac{1}{x^6}+\frac{1}{x^4}+\frac{1}{x^2}+1} = \frac{0}{1} = \boxed{0}.$$

Question 11 (6 points)

Find the derivative of

$$f(x) = x \sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x^3}\right)$$

$$f'(x) = x\left[\sin\left(\frac{1}{x^2}\right)\cdot\left(-\frac{3}{x^4}\cdot -\sin\left(\frac{1}{x^3}\right)\right) + \left(-\frac{2}{x^3}\cdot\cos\left(\frac{1}{x^2}\right)\cdot\cos\left(\frac{1}{x^3}\right)\right)\right]$$

$$+ \sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x^3}\right)$$

Bonus: What is $\lim_{x\to 0} f(x)$?

$$-1 \leq \sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x^3}\right) \leq 1$$

$$-|x| \leq x\sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x^3}\right) \leq |x|$$

Since $\lim_{x\to 0}|x| = 0$, $\lim_{x\to 0} -|x| = 0$, we have

$$\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)\cos\left(\frac{1}{x^3}\right) = \boxed{0}$$

Question 12 (6 points)

Find the derivative of

$$f(x) = 3^{\ln(x)} + 4^{\sqrt{x}} + 5^{\sin(x-1)}$$

$$f'(x) = \frac{\ln 3 \cdot 3^{\ln(x)}}{x} + \frac{\ln 4 \cdot 4^{\sqrt{x}}}{2\sqrt{x}} + \ln 5 \cdot 5^{\sin(x-1)} \cdot \cos(x-1)$$

Bonus: What is the slope of the tangent line to f at $x = 1$?

$$f'(1) = \frac{\ln 3 \cdot 3^{\overset{0}{\ln(1)}}}{1} + \frac{\ln 4 \cdot 4^{\sqrt{1}}}{2\sqrt{1}} + \ln 5 \cdot \overset{1}{5^{\sin(0)}} \cdot \overset{1}{\cos(0)}$$

$$= \ln 3 + 2\ln 4 + \ln 5$$

$$= \ln 3 + \ln 16 + \ln 5$$

$$= \boxed{\ln 240}$$

Question 13 (6 points)

Find the derivative of

$$f(x) = \sec(\arctan(e^{\tan(x)}))$$

$$f'(x) = \sec(\arctan(e^{\tan x}))\tan(\arctan(e^{\tan x})) \cdot \frac{1}{1+(e^{\tan x})^2} e^{\tan x} \sec^2 x .$$

Bonus: What is $f(0)$?

$$f(0) = \sec(\arctan(e^{\tan(0)}))$$
$$= \sec(\arctan(e^0))$$
$$= \sec(\arctan(1)) = \sec(\tfrac{\pi}{4})$$
$$= \frac{1}{\cos(\frac{\pi}{4})} = \frac{1}{\frac{\sqrt{2}}{2}} = \boxed{\sqrt{2}}$$

Question 14 (6 points)

Find the derivative of

$$f(x) = \sec(\cancel{\operatorname{arcsec}}(\sin(\cancel{\arcsin}(\tan(\cancel{\arctan}(x^2))))))$$

$$f(x) = x^2$$
$$f'(x) = 2x$$

Bonus: What is $f'''(10)$?

$$f''(x) = 2$$
$$f'''(x) = 0 .$$

$$\boxed{f'''(10) = 0}$$

Question 15 (8 points)

Find the derivative of

$$f(x) = \sqrt{\frac{x^2}{1+\frac{x}{1+\frac{x}{2x+1}}}}$$

$$f(x) = \sqrt{\frac{x^2}{1+\frac{x}{\frac{3x+1}{2x+1}}}} = \sqrt{\frac{x^2}{1+\frac{2x^2+x}{3x+1}}}$$

$$= \sqrt{\frac{x^2(3x+1)}{2x^2+4x+1}} = \sqrt{\frac{3x^3+x^2}{2x^2+4x+1}}$$

Bonus: What are the zeros of the function f?

$x^2=0$

$\boxed{x=0}$

$$f'(x) = \frac{1}{2\sqrt{\frac{x^2(3x+1)}{2x^2+4x+1}}}\left(\frac{(2x^2+4x+1)(9x^2+2x)-(3x^3+x^2)(4x+4)}{(2x^2+4x+1)^2}\right)$$

Question 16 (8 points)

Find the derivative of

$$f(x) = \frac{\ln(\ln(x))+\ln(x)}{e^{\sqrt{x+3}}+\sin(\sqrt{x+3})}$$

$$f'(x) = \frac{\left(e^{\sqrt{x+3}}+\sin(\sqrt{x+3})\right)\left(\frac{1}{x\ln x}+\frac{1}{x}\right)-(\ln(\ln x)+\ln x)\left(\frac{e^{\sqrt{x+3}}}{2\sqrt{x+3}}+\frac{\cos(\sqrt{x+3})}{2\sqrt{x+3}}\right)}{\left(e^{\sqrt{x+3}}+\sin(\sqrt{x+3})\right)^2}$$

Bonus: What is $\lim_{x\to\infty} f(x)$?

$\lim_{x\to\infty} f(x)=0$ since $e^{\sqrt{x+3}} \to \infty$ in denominator and $\ln x$ grows slower than a constant times x.

Question 17 (8 points)

Find the derivative of

$$f(x) = \arctan(\arctan(\arctan(x))) + e^{-x}$$

$$f'(x) = \frac{1}{1+(\arctan(\arctan(x)))^2}\left(\frac{1}{1+(\arctan x)^2}\right)\left(\frac{1}{1+x^2}\right) - e^{-x}.$$

Bonus: What is the equation of the tangent line to f at $x = 0$?

$$f(0) = \arctan(\arctan(\arctan(0))) + e^0 = 0 + 1 = 1$$

$$f'(0) = \frac{1}{1+0^2}\left(\frac{1}{1+0^2}\right)\left(\frac{1}{1+0^2}\right) - e^0 = 0 \qquad \boxed{y=1}$$

Question 18 (8 points)

Find the derivative of

$$f(x) = \arctan(\sin(e^x - 1)) + \tan(\ln(\arcsin(x) + 1))$$

$$f'(x) = \frac{1}{1+(\sin(e^x-1))^2} e^x \cos(e^x - 1)$$
$$+ \sec^2(\ln(\arcsin(x)+1)) \cdot \frac{1}{\arcsin x + 1} \cdot \frac{1}{\sqrt{1-x^2}}$$

Bonus: What is $f(0)$?

$$f(0) = \arctan(\sin(0)) + \tan(\ln(1))$$
$$= \arctan(0) + \tan(0) = \boxed{0}.$$

Question 19 (10 points)

Find the derivative of

$$f(x) = \sqrt{\arcsin(\ln(x)) + \sqrt{\arccos(\ln(x)) + \sqrt{\arctan(\ln(x))}}}$$

$$f'(x) = \frac{1}{2\sqrt{\arcsin(\ln x) + \sqrt{\arccos(\ln x) + \sqrt{\arctan(\ln x)}}}} \cdot \left\{ \frac{1}{\sqrt{1-(\ln x)^2}}\frac{1}{x} + \frac{1}{2\sqrt{\arccos(\ln x) + \sqrt{\arctan(\ln x)}}}\left[\frac{-1}{\sqrt{1-(\ln x)^2}}\frac{1}{x} + \frac{1}{2\sqrt{\arctan(\ln x)}}\left(\frac{1}{1+(\ln x)^2}\frac{1}{x}\right)\right]\right\}$$

Bonus: What is $f(e)$?

$$f(e) = \sqrt{\arcsin(1) + \sqrt{\arccos(1) + \sqrt{\arctan(1)}}}$$

$$= \sqrt{\frac{\pi}{2} + \sqrt{0 + \sqrt{\frac{\pi}{4}}}} = \boxed{\sqrt{\frac{\pi}{2} + \sqrt{\sqrt{\frac{\pi}{4}}}}}$$

Question 20 (10 points)

Find the derivative of

$$f(x) = e^{\overbrace{\left(\frac{\sin(x)}{x} + \frac{\sin^2(x)}{x^2} + \frac{\sin^3(x)}{x^3}\right)}^{\left(\frac{\sin x}{x}\right) + \left(\frac{\sin x}{x}\right)^2 + \left(\frac{\sin x}{x}\right)^3}} \cdot \ln\left(1 + \frac{1}{\arctan(e^x)} + \frac{1}{\arctan(e^{2x})}\right)$$

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}$$

$$f'(x) = \left(\frac{x\cos x - \sin x}{x^2}\right)\left(1 + \frac{2\sin x}{x} + 3\left(\frac{\sin x}{x}\right)\right)^2 f(x)$$

$$+ \frac{e^{\frac{\sin x}{x} + \frac{\sin^2 x}{x^2} + \frac{\sin^3 x}{x^3}}}{1 + \frac{1}{\arctan(e^x)} + \frac{1}{\arctan(e^{2x})}}\left[-\frac{1}{(\arctan(e^x))^2}\frac{e^x}{1+e^{2x}} - \frac{1}{(\arctan(e^{2x}))^2}\frac{2e^{2x}}{1+e^{4x}}\right]$$

$$\frac{d}{dx}\left(\frac{1}{\arctan(e^x)}\right) = -\frac{1}{(\arctan(e^x))^2}\frac{1}{1+(e^x)^2}e^x$$

$$\frac{d}{dx}\left(\frac{1}{\arctan(e^{2x})}\right) = -\frac{1}{(\arctan(e^{2x}))^2}\frac{1}{1+e^{4x}}2e^{2x}$$

Bonus: What is $\lim_{x\to 0} f(x)$, if it exists?

Use $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

$$\lim_{x\to 0}\frac{\sin^2 x}{x^2} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2 = 1^2 = 1$$

$$\lim_{x\to 0}\frac{\sin^3 x}{x^3} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3 = 1^3 = 1.$$

$$\frac{1}{\arctan(1)} = \frac{4}{\pi}.$$

So $\boxed{\lim_{x\to 0} f(x) = e^3\ln\left(1 + \frac{8}{\pi}\right)}$