Math 365C: Real Analysis I

The University of Texas at Austin, Spring 2021

Official times: MWF 1-2PM CST
Media: *Canvas, Zoom, Piazza*
(see "Course delivery and resources")

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Office hours: schedule on *Canvas*

This course is an introduction to analysis. Analysis, together with algebra and topology, form the central core of modern pure mathematics. Analysis is also indispensable to the applied and computational mathematician. Beginning with the notion of limit from calculus and continuing with ideas about convergence and the concept of function that arose with the description of heat flow using Fourier series, analysis is primarily concerned with *infinite processes*, the study of spaces where these processes act and the application of differential and integral methods.

Specifically in this course, we will have three principle focuses. We will rigorously develop the operations and concepts of single-variable calculus (limits, continuity, derivative, Taylor series, Riemann integral). We will study abstract metric spaces (capturing the general notion of distance). Time permitting, we will sketch the beginnings of functional analysis (where functions are regarded as points in a space).

At risk of sounding melodramatic, learning real analysis is a *rite of passage* for undergraduate math majors (both those with pure and applied interests). At the same time, the present course is important to theoretically minded students in adjacent disciplines, such as physics, computer science, statistics, electrical engineering, finance and economics. You might also find analysis good for your health: promoting rigorous and abstract thinking, and empowering you to turn pictures into proofs.

Textbook

The course textbook is Walter Rudin's *Principles of Mathematical Analysis*, 3rd ed. We will cover the first five chapters and the seventh chapter. For Riemann integration, we will follow Chapter 7 of Stephen Abbott's *Understanding Analysis*, 2nd ed.

As for **optional** supplementary reading, *Real Mathematical Analysis*, 2nd ed. by Charles Pugh is nice for intuition and for having many pictures. *Introduction to Metric and Topological Spaces*, 2nd ed. by Wilson Sutherland goes beyond our scope, but certain parts might be helpful for understanding metric spaces.

Course delivery and resources

Our course will be delivered using the following resources and structure:

- **Lectures** will be *asynchronous*: they will be prerecorded and posted on *Panopto* integrated in *Canvas* (typically not later than the corresponding official lecture time MWF 1-2PM CST). Handwritten notes produced during the lectures will be saved as PDFs and also uploaded to *Canvas*.

- **Exam review problem-solving sessions** will be *synchronous*, occurring live on *Zoom*, and will include students working in breakout rooms and some presenting to the class. Also, the first informal organizational lecture on Wednesday, January 20 will occur live on *Zoom*.

- **Office hours** in this course will be plentiful: most weeks we will have six office hours, typically at least one hour Tuesday–Friday. The instructor, teaching assistant and course consultant will each generally offer two hours per week over *Zoom* (see *Canvas* for a schedule). Please take advantage of office hours.

- **Discussion boards** will be set up on *Piazza* integrated with *Canvas*. **Course-related questions should be posted on *Piazza* rather than emailed to course staff** (with the exception of questions on personal matters). Course staff will answer questions on *Piazza* and moderate discussion there. Students are encouraged to try their hand in answering some of their classmates' questions on *Piazza*, as this is a great way to learn yourself. Note that making mistakes in *Piazza* is absolutely acceptable; we personally think it is also a good way to learn. For students falling very near boundaries between letter grades, *Piazza* participation *might* be taken into discretionary consideration; however, for this to help boost a boundary case, it *must* be the case that the student has posted to *Piazza* **non-anonymously** (visible to other students) **exclusively**.

Legal notes: course materials should not be distributed outside the course. The organizational meeting and some problem-solving sessions may be recorded. Class recordings are reserved only for students in this class and are protected under FERPA. The recordings should not be shared outside the class in any form. Violation of this restriction by a student could lead to Student Misconduct proceedings.

Grading

Numerical scores will be computed per this breakdown:

- Homework: 20% (lowest two dropped)
- Midterm I: 20%
- Midterm II: 20%
- Final: 40%

Letter grades with plus and minuses will then be determined according to a curve. The end distribution of letters will be similar to previous iterations of Math 365C.

Homework

There will be roughly weekly homework assignments. They will be posted on *Gradescope* integrated with *Canvas*. Submission is through *Gradescope*

Altogether there will be nine homeworks. Each student's lowest two homework grades will be dropped. Late homework will not be accepted. You are encouraged to discuss problems with classmates, but you must write all solutions that you submit in your own words yourself. If you wish to typeset your solutions in LaTeX, you may find the online editor *Overleaf* helpful. You should not search for answers to the problems online, particularly if you wish to learn the material adequately for success on the exams. Homeworks will be graded for mathematical correctness as well as clarity of presentation; if we cannot make any sense of parts of your written proof, then this poor communication must result in lost points. Since we also want you to internalize important results from the textbook, on some of the homeworks, in place of some problem solving, you will be asked to submit a short video explaining in your own words certain key arguments or concepts from the course (details on the videos to follow).

Exams

There will be two midterms and one final exam. Details on the formats and mechanism of proctoring will be announced in due course. At the start of the semester, please reserve our planned dates (as makeup exams generally cannot be arranged).

- The first midterm will occur on **Friday, March 5**. It will cover lectures 2 – 11, 16–17 and homeworks 1 – 4 (real numbers, metric spaces, sequences).
- The second midterm will occur on **Friday, April 16**. It will cover lectures 21 – 30 and homeworks 5 – 8 (series, continuity, differentiability).
- The final will be open-course materials, untimed, take-home taking place during **morning, Wednesday, May 12 – 23:59pm, Monday, May 17**. The final will cover all parts of the course, but with emphasis on lectures 34 – 42 and the last homeworks (integration, sequences and series of functions).

Several live exam review sessions will be offered. You will receive a list of exercises on which you will work in *Zoom* breakout rooms. Students may present their solutions, with our moderation or help. It should go without saying, but we emphasize that the review sessions cannot possibly touch on every examinable topic for which you are responsible. This holds true for the final as well as the midterms.

Consistent with the homeworks, exams shall be graded for mathematical correctness as well as clarity of presentation.

Breakdown of lectures

A tentative schedule for lecture content is shown on the last page.

Course announcements

Important course announcements will typically be double-posted, on *Canvas Announcements* and on *Piazza*. In any case, students are responsible for making themselves aware of important course announcements.

Accommodations

The University of Texas provides, upon request, appropriate academic accommodations for qualified students with disabilities. For more information, contact Services for Students with Disabilities at 512-471-6259 or ssd@austin.utexas.edu

Academic integrity

The University of Texas holds you to the following Standards of Conduct:
https://deanofstudents.utexas.edu/conduct/standardsofconduct.php.
Violations of these Standards shall be treated seriously, and punished appropriately.

Blanket caveat

This syllabus is subject to change. Students are responsible for making themselves aware of syllabus changes announced in *Canvas Announcements* and *Piazza*.

Lecture #	Date	Topics	Text
1 (informal, live)	Wed, Jan 20	everyone says hi, organization of the course, what's a real number anyway?	–
2	Fri, Jan 22	$\mathbb{R}$ as ordered field with least-upper-bound property, archimedean property	Chapter 1
3	Mon, Jan 25	Euclidean space, Dedekind cuts (sketch)	Chapter 1
4	Wed, Jan 27	countable sets	Chapter 2
5	Fri, Jan 29	metric space: basic definitions & examples	Chapter 2
6	Mon, Feb 1	metric space: basic definitions & examples	Chapter 2
7 (drop date)	Wed, Feb 3	compact sets	Chapter 2
8	Fri, Feb 5	compact sets	Chapter 2
9	Mon, Feb 8	connected sets	Chapter 2
10	Wed, Feb 10	convergent sequences in metric spaces	Chapter 3
11	Fri, Feb 12	Cauchy sequences in metric spaces, complete metric spaces	Chapter 3
12	Mon, Feb 15	**STORM**	–
13	Wed, Feb 17	**STORM**	–
14	Fri, Feb 19	**STORM**	–
15	Mon, Feb 22	**STORM**	–
16	Wed, Feb 24	sequential compactness, completeness	Chapter 3
17	Fri, Feb 26	numerical examples, lim sup & lim inf	Chapter 3
18 (midterm review, live)	Mon, Mar 1	problems session on *Zoom*	–
19 (midterm review, live)	Wed, Mar 3	problems session on *Zoom*	–
20 (no lecture, exam)	Fri, Mar 5	**MIDTERM I** (covering lectures 2-11, 16-17)	–
21	Mon, Mar 8	**BREAK**	–
22	Wed, Mar 10	root & ratio tests	Chapter 3
23	Fri, Mar 12	power series, absolute convergence	Chapter 3
–	Mon, Mar 15 – Fri, Mar 19	**SPRING BREAK**	–
24	Mon, Mar 22	addition, multiplication, rearrangment of series	Chapter 3
25	Wed, Mar 24	continuity of functions on metric spaces	Chapter 4
26	Fri, Mar 26	continuity & compactness	Chapter 4
27	Mon, Mar 29	uniform continuity, continuity & connectedness	Chapter 4
28	Wed, Mar 31	examples, one-sided limits, limits at ∞	Chapter 4
29	Fri, Apr 2	derivative of real-valued function (basics)	Chapter 5
30	Mon, Apr 5	local extrema, Rolle's theorem, mean value theorem	Chapter 5
31	Wed, Apr 7	continuity of derivatives, L'Hôspital's rule, higher derivatives	Chapter 5
32	Fri, Apr 9	Taylor's theorem, remainder term, examples	Chapter 5
33 (midterm review, live)	Mon, Apr 12	problems session on *Zoom*	–
34 (midterm review, live)	Wed, Apr 14	problems session on *Zoom*	–
35 (no lecture, exam)	Fri, Apr 16	**MIDTERM II** (covering lectures 17-28)	–
36	Mon, Apr 19	upper & lower Riemann integrals, construction of Riemann-Stieltjes integral	Abbott
37	Wed, Apr 21	refinements, continuous functions are integrable	Abbott
38	Fri, Apr 23	monotonic functions, bounded functions with finitely many discontinuities	Abbott
39	Mon, Apr 26	first properties of Riemann-Stieltjes integral (e.g., linearity, supremum bound), change of variable	Abbott
40	Wed, Apr 28	fundamental theorem of calculus, integration by parts	Abbott
41	Fri, Apr 30	pointwise limit of functions, swapping order of limits (counterexamples)	Chapter 7
42	Mon, May 3	uniform convergence, examples	Chapter 7
43	Wed, May 5	uniform convergence & continuity, uniform convergence & integration	Chapter 7
44 (final review, live)	Fri, May 7	problems session on *Zoom*	–
–	date TBA	**FINAL** (covering all lectures, with emphasis on lectures 32-43)	–

Real Analysis: *Homework 1*

Due: 23:59 CST, January 30, 2021

1. Let $S = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$. Without appealing to the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, prove from first principles that if $x \in S$, then there exists $y \in S$ with $y > x$. *Hint*: let $y = x + \varepsilon$ where $\varepsilon \in \mathbb{Q}_{>0}$. Figure out how small to set ε so that $y^2 < 2$.

2. Prove there does not exist an order on the complex field $\mathbb{C}$ making $\mathbb{C}$ into an ordered field. *Hint:* $i^2 = -1$.

3. For $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$ in Euclidean space $\mathbb{R}^k$, declare $\mathbf{x} \prec_{\text{lex}} \mathbf{y}$ if the leftmost nonzero entry of $\mathbf{y} - \mathbf{x}$ is positive. Prove $(\mathbb{R}^k, \prec_{\text{lex}})$ is an ordered set. One calls $\prec_{\text{lex}}$ the *lexicographic order*. Does $(\mathbb{R}^k, \prec_{\text{lex}})$ have the least-upper-bound (LUB) property?

4. Let $(S, <)$ be an ordered set. Let $E \subseteq S$. We say that E is *bounded below* and β is a *lower bound* for E if $\beta \in S$ and $\beta \leq x$ for all $x \in E$. We say that $\alpha \in S$ is a *greatest lower bound* for E if α is a lower bound for E and $\alpha \geq \beta$ whenever $\beta \in S$ is also a lower bound for E.

   (i) Show that when a greatest lower bound for E exists, it is unique.

   The greatest lower bound for E is also called the *infimum* of E, denoted $\inf E$. We say that the ordered set $(S, <)$ has the *greatest-lower-bound (GLB) property* if: for all subsets $E \subseteq S$ that are nonempty and bounded below, $\inf E$ exists.

   (ii) Prove that $(S, <)$ has the GLB property if it has the LUB property.

   Remark: the converse holds too, so that GLB and LUB are equivalent.

5. Let the function $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 + x + 1$. By considering $E := \{x \in \mathbb{R} : f(x) < 0\} \subseteq \mathbb{R}$ and using the least-upper-bound property for $\mathbb{R}$, prove that f has a *root* in $\mathbb{R}$ (i.e., there exists $r \in \mathbb{R}$ such that $f(r) = 0$). Please explain carefully. *Hint:* adapt the argument I used in lectures to show a positive n-th root of a positive real number exists in $\mathbb{R}$.

Real Analysis: *Homework 2*

Due: 23:59 CST, February 7, 2021

1. Given a set T, let $\mathcal{P}(T)$ denote the set of all subsets of T. One calls $\mathcal{P}(T)$ the *power set* of T. For example, $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$. Consider the following subset of the power set of the natural numbers:

$$\mathcal{I} := \{S \in \mathcal{P}(\mathbb{N}) : S \text{ is infinite}\}.$$

Exhibit an explicit bijection $f : \mathcal{I} \to \mathbb{R}$, or prove a bijection doesn't exist. *Hint:* first see if you can biject $\mathcal{I}$ with the interval $(0,1] \subseteq \mathbb{R}$ using binary strings. If so, can you biject $(0,1]$ with $(0,1)$, and then $(0,1)$ with $\mathbb{R}$?

2. I proved in lectures that the set $\mathbb{R}$ is uncountable. Prove nonetheless that the metric space $\mathbb{R}$ (with its usual metric) is *second-countable*. A metric space (M,d) is said to be second-countable if there exists a countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open sets $U_i \subseteq M$ such that any open set in M may be written as a union of a subcollection of $\mathcal{U}$. *Hint:* $\mathbb{Q}$ is countable.

3. For $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$, define

$$d_1(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{k} |x_i - y_i|.$$

(i) Prove that d_1 is a metric on $\mathbb{R}^k$.
(ii) People sometimes call d_1 the *Manhattan metric*. By means of a sketch in $\mathbb{R}^2$, explain why this is reasonable. *Hint:* If you don't know, the streets in Manhattan are arranged in a grid.
(iii) Write d_2 for the metric on $\mathbb{R}^k$ induced by the Euclidean inner product, $d_2(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_2 = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$. Prove that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$,

$$\tfrac{1}{\sqrt{k}}\, d_1(\mathbf{x}, \mathbf{y}) \;\leq\; d_2(\mathbf{x}, \mathbf{y}) \;\leq\; d_1(\mathbf{x}, \mathbf{y}).$$

Hint: for one of these inequalities, use Cauchy-Schwarz.
(iv) Deduce that $(\mathbb{R}^k, d_1)$ and $(\mathbb{R}^k, d_2)$ have the same open sets. We can express this by saying that d_1 and d_2 are *topologically equivalent*.
(v) In $\mathbb{R}^2$, sketch the open ball centered at the origin of unit radius with respect to d_1, and likewise with respect to d_2.

4. Let M be a metric space with metric d. For $x \in M$ and $A \subseteq M$, define

$$d(x, A) := \inf_{a \in A} d(x, a).$$

(Recall from HW1 that 'inf' denotes infimum, or greatest lower bound.)
(i) Show that if y is another point of M,

$$d(y, A) \leq d(y, x) + d(x, A).$$

(ii) Fill in the blanks: $d(x, A) = 0$ if and only if x is a point in A or a ____________ of A if and only if x is a point in the ____________ of A.

5. Fix a prime number p. Given $n_1, n_2 \in \mathbb{Z}$, define $d_p(n_1, n_2)$ to be p^{-r} where p^r ($r \in \mathbb{Z}_{\geq 0}$) is the largest power of p that divides $n_1 - n_2$ if $n_1 \neq n_2$, and define $d_p(n_1, n_2)$ to be 0 if $n_1 = n_2$. For example, $d_2(5, 17) = 2^{-2}$ because 2^2 divides $5 - 17 = -12$ but 2^3 does not. Prove that d_p satisfies the *ultrametric inequality*:

$$d_p(n_1, n_3) \leq \max\left(d_p(n_1, n_2), d_p(n_2, n_3)\right) \quad \forall n_1, n_2, n_3 \in \mathbb{Z}.$$

Deduce that d_p defines a metric on $\mathbb{Z}$. This is called the *p-adic metric.*

Real Analysis: *Homework 3*

Due: 23:59 CST, February 27, 2021

1. Give an example of a metric space M which has a closed ball of radius 1.001 which contains 100 *disjoint* closed balls of radius one. *Hint*: tweak the discrete metric.

2. Record a video in which you discuss the following capstone result:

   A subset of Euclidean space $\mathbb{R}^k$ is compact if and only if it is closed and bounded.

   In your own words, you should explain the meaning of this result, outline the proof in reasonable but not necessarily total detail, and provide an example as well as a non-example. The video should include mathematical formulas as my lectures do, which you can write out by hand or in slides. Please keep the video under six minutes. **See *Canvas Announcements* or *Piazza* for the instructions on submitting your video, which must be done separately to the submission of this homework.**

3. We say a metric space M has the *Heine-Borel property* if all closed bounded subsets of M are compact. By Question 1, Euclidean space (with its usual metric) has the Heine-Borel property. Prove that $\mathbb{Q}$ (with the subspace metric induced from $\mathbb{R}$) does not have Heine-Borel property.

4. Let M be a metric space. If $E \subseteq M$ is a maximal (under inclusion) connected subset of M, we say that E is a *connected component* of M. This means: E is connected and if $E' \subseteq M$ is also connected with $E \subseteq E'$, then $E' = E$.

   (i) Prove that the connected components of any metric space partition the metric space.

   (ii) Find a compact subset $K \subseteq \mathbb{R}$ for which $\mathbb{R} \setminus K$ has infinitely many connected components. *Hint*: remember the fact from lectures that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

5. Given two metric spaces (M, d_M) and (N, d_N), consider the cartesian product $M \times N$ and define the function $d_{M\times N} : (M \times N) \times (M \times N) \to \mathbb{R}$ by

$$d_{M\times N}((x_1, y_1), (x_2, y_2)) := \sqrt{d_M(x_1, x_2)^2 + d_N(y_1, y_2)^2}$$

for all $(x_1, y_1), (x_2, y_2) \in M \times N$.

(i) Prove that $d_{M\times N}$ is a metric on $M \times N$. We call this a *product metric*.

(ii) Let W be an open set in $M \times N$. Prove that $W = \bigcup_\alpha U_\alpha \times V_\alpha$ for some open sets U_α in M and some open sets V_α in N. We say $M \times N$ has the *product topology*.

(iii) Prove that if (M, d_M) and (N, d_N) are both compact, then so is $(M \times N, d_{M\times N})$. *Hint*: Consider an open cover of $M \times N$. You need to show there exists a finite subcover. By part (ii), argue that you may assume the open cover is of the form $\{U_\alpha \times V_\alpha\}_\alpha$ where U_α and V_α are open sets in M and N, respectively. Now for each $x \in M$, consider $\{V_\alpha : x \in U_\alpha\}$. Argue that this is an open cover of N. Now invoke compactness of N. Now figure out how to use compactness of M. *Alternative Hint*: If you prefer not to work with the open cover definition of compactness here, you can instead do this question using the characterization that a set K is compact if and only if every infinite subset of K has a limit point in K.

(iv) Prove that if (M, d_M) and (N, d_N) are both connected, then so is $(M \times N, d_{M\times N})$. *Hint*: Suppose $M \times N = W_1 \sqcup W_2$ (disjoint union) for open sets $W_1, W_2 \subseteq M \times N$ with $W_1 \neq \emptyset$. You need to show $W_2 = \emptyset$. Use part (ii) and reason somewhat similarly to part (iii), first hinted approach.

Real Analysis: *Homework 4*

Due: 23:59 CST, March 2, 2021

1. Directly using the $\epsilon - N$ definition of the limit of a sequence, determine

$$\lim_{n\to\infty} \sqrt{n^2+3n} - n.$$

2. Find the limit inferior and superior of the sequence (s_n) in $\mathbb{R}$ given by:

$$s_1 = 0; \qquad s_{2m} := \frac{s_{2m-1}}{2}; \qquad s_{2m+1} := \frac{1}{2} + s_{2m},$$

where the above recursion holds for all $m \in \mathbb{N}$.

3. Let (x_n) be a sequence in a compact metric space M. Suppose (x_n) does not converge. Prove that it has two convergent subsequences with different limits.

4. Suppose $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence in a metric space M such that there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ converging to $p \in M$. Prove that the whole sequence $(a_n)_{n=1}^{\infty}$ converges to p.

5. Let M denote the set of all bounded real sequences $x = (x_n)$. For $x, y \in M$, define $d(x, y) := \sup_n |x_n - y_n|$.

   (i) Prove that (M, d) is a metric space. We call d the *supremum metric.*

   (ii) Prove that (M, d) is complete.

   (iii) Exhibit a bounded sequence in M with no convergent subsequence.

Real Analysis: *Homework 5*

Due: 23:59 CDT, March 26, 2021

1. In this exercise, we work through basic properties of limit superiors and limit inferiors. Let $(a_n)_{n=1}^{\infty}$ be a real sequence.

   (a) Prove that the maximal/minimal subsequential limit definition of

   $$\limsup_{n\to\infty} a_n \quad \text{and} \quad \liminf_{n\to\infty} a_n$$

   given in lectures and the textbook is equivalent to the following:

   $$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{m\geq n} a_m\right) \quad \text{and} \quad \liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{m\geq n} a_m\right).$$

   (b) Suppose $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are such that

   $$\alpha > \beta, \quad \beta = \limsup_{n\to\infty} a_n, \quad \gamma = \liminf_{n\to\infty} a_n, \quad \gamma > \delta.$$

   Prove there exists $N \in \mathbb{N}$ such that

   $$\alpha > a_n > \delta \quad \text{for all } n \geq N.$$

2. Decide (with justification) whether each of the following series converges or diverges. Where not indicated, Σ is over n from 1 to ∞.

   (a) $\sum_{n=5}^{\infty} \frac{1}{2^n - n^2}$

   (b) **[Extra Credit]** $\sum \frac{\sin(n)}{n}$ *Hint:* You may assume Theorem 3.42 from the book.

   (c) $\sum(\sqrt{n+1} - \sqrt{n})$

   (d) $\sum \frac{\sqrt{n+1}-\sqrt{n}}{n}$

   (e) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \dots$

   (f) $\sum(\sqrt[n]{n} - 1)^n$

(g) $\sum \big(n!/n^n\big)$

(h) $\sum_{n=2}^{\infty} \big(1/\log(n)^{\log(n)}\big)$

(i) $\sum \log(\frac{n+1}{n})$

(j) $\sum \frac{1}{1+z^n}$ where $z \in \mathbb{R}$ is fixed (your answer may depend on z)

3. For each of the following power series in $z \in \mathbb{R}$, determine (with justification) the radius of convergence R and whether the series converges absolutely, converges non-absolutely or diverges at the boundary points $z = \pm R$ (in cases when R is finite). Here each Σ is over n from 1 to ∞. *Hint:* If you try to apply the lim sup expression for the radius of convergence, remember that the limit superior of a sequence simply equals the limit when the limit exists.

   (a) $\sum n^3 z^n$

   (b) $\sum \frac{2^n}{n!} z^n$

   (c) $\sum \frac{2^n}{n^2} z^n$

   (d) $\sum \frac{n^3}{3^n} z^n$

4. Consider the series Σa_n given by

$$\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{8} + \tfrac{1}{8} - \tfrac{1}{8} + \tfrac{1}{8} - \tfrac{1}{8} + \tfrac{1}{8} - \tfrac{1}{8} + \tfrac{1}{16} - \tfrac{1}{16} + \dots$$

   (a) Explain why Σa_n converges non-absolutely.

   (b) Discuss (with justification) an explicit rearrangement $\Sigma a'_n$ of Σa_n such that

$$\limsup_{n\to\infty} s'_n = 100 \qquad \text{and} \qquad \liminf_{n\to\infty} s'_n = -1,$$

   where (s'_n) is the sequence of partial sums of the rearrangement $\sum a'_n$.

5. For a sequence $(b_n)_{n=1}^{\infty} \subseteq \mathbb{R}$, the *infinite product*

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \dots$$

is defined to be the limit of *partial products*,

$$\prod_{n=1}^{\infty} b_n := \lim_{n\to\infty} p_N \quad \text{where} \quad p_N := \prod_{n=1}^{N} b_n = b_1 b_2 \dots b_N,$$

when this limit exists (else the infinite product is divergent).

(a) Consider infinite products in which each factor is at least 1, i.e.,

$$\prod_{n=1}^{\infty}(1+a_n) \quad \text{where } \; a_n \in \mathbb{R}_{\geq 0}.$$

Prove this infinite product converges if and only if the series $\Sigma_{n=1}^{\infty} a_n$ converges. *Hint*: for one direction, you may assume without proof that $\log_3(1+x) \leq x$ for all $x \geq 0$.

(b) The following formula for π was discovered in the 1600s:

$$\pi \;=\; 2\prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1} \;=\; 2\left(\frac{2}{1}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{4}{5}\right)\left(\frac{6}{5}\cdot\frac{6}{7}\right)\left(\frac{8}{7}\cdot\frac{8}{9}\right)\cdots$$

Using part (a), prove that the infinite product indeed converges (you need not verify the limit is π). Using a computer or calculator, compute the partial products $2\prod_{n=1}^{N}\frac{4n^2}{4n^2-1}$ to three digits after the decimal for $N = 1, \ldots, 10$.

6. **[Extra Credit]** In this bonus exercise, we consider attaching a limit definition to a *double* series.

Let $(a_{m,n} : m, n \in \mathbb{N})$ be a doubly indexed infinite array of real numbers.

- For each m, we call $\sum_{n=1}^{\infty} a_{m,n} := \lim_{N\to\infty}\sum_{n=1}^{N} a_{m,n}$ the *m-th row series*.
- For each n, we call $\sum_{m=1}^{\infty} a_{m,n} := \lim_{M\to\infty}\sum_{m=1}^{M} a_{m,n}$ the *n-th column series*.
- We call $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} := \lim_{M\to\infty}\sum_{m=1}^{M}\left(\lim_{N\to\infty}\sum_{n=1}^{N} a_{m,n}\right)$ the *row-first iterated series*.
- We call $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n} := \lim_{N\to\infty}\sum_{n=1}^{N}\left(\lim_{M\to\infty}\sum_{m=1}^{M} a_{m,n}\right)$ the *column-first iterated series*.
- We call $\sum_{m,n=1}^{\infty} a_{m,n} := \lim_{P\to\infty}\sum_{m,n=1}^{P} a_{m,n}$ the *double series*.

We say that an iterated series converges if and only if each inner limit converges and the series of such (the outer limit) also converges.

(a) Can you find $(a_{m,n})$ for which each row series diverges to $+\infty$, each column series diverges to $-\infty$, yet the double series converges?

(b) Prove that if the iterated series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{m,n}|$ converges, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}, \qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{m,n}, \qquad \sum_{m,n=1}^{\infty} a_{m,n}$$

all converge to the same real number.

Real Analysis: *Homework 6*

Due: 23:59 CDT, April 4, 2021

`Unless stated otherwise,` $\mathbb{R}$ `and` $\mathbb{R}^n$ `are equipped with their usual Euclidean metrics.`

1. In this exercise, we get hands-on experience with verifying if a function is continuous or not.

   (a) Show directly from the $\varepsilon - \delta$ definition of continuity that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{|x|}$ is continuous.

   (b) Let X be any nonempty set. Recall the discrete metric on X is given by

   $$d(x, x') = \begin{cases} 1 & \text{if } x \neq x' \\ 0 & \text{if } x = x'. \end{cases}$$

   What are the open sets in (X, d)? Which functions $f : X \to \mathbb{R}$ are continuous? Which functions $f : \mathbb{R} \to X$ are continuous?

   (c) Every rational number $x \in \mathbb{Q}$ can be written uniquely as p/q where $p \in \mathbb{Z}$, $q \in \mathbb{Z}_{>0}$, and p and q share no common factor (we are just writing x in lowest terms). Define the function $f : \mathbb{R} \to \mathbb{R}$ by

   $$f(x) = \begin{cases} 1/q & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Prove f is continuous at every irrational point, and discontinuous at every rational point.

2. A (real multivariate) polynomial is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form $f(\mathbf{x}) = \sum_{\alpha \in \mathcal{A}} c_\alpha \mathbf{x}^\alpha$ where $\mathcal{A} \subseteq (\mathbb{Z}_{\geqslant 0})^n$ is a finite subset (*the set of exponents*), $c_\alpha \in \mathbb{R}$ are scalars (*the coefficients*) and $\mathbf{x}^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geqslant 0})^n$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ (*the monomials*). For example, $f(x_1, x_2, x_3) = x_1^3 - 10x_1x_2x_3 + 0.5x_1^2 - x_2 + 1$ is a polynomial function $\mathbb{R}^3 \to \mathbb{R}$.

   (a) By using propositions from lecture together with induction, prove that every polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

   (b) Let $f_1, \dots, f_k$ be polynomial functions $\mathbb{R}^n \to \mathbb{R}$. Prove that the *set of common zeros*

   $$\mathcal{Z}(f_1, \dots, f_k) := \{\mathbf{x} \in \mathbb{R}^n : f_1(\mathbf{x}) = \dots = f_k(\mathbf{x}) = 0\}$$

is a closed subset of $\mathbb{R}^n$. One also calls this set the *solution set* to the polynomial system defined by $f_1, \ldots, f_k$ (or in other language, the *real algebraic variety* cut out by $f_1, \ldots, f_k$).

3. Let $f : (X, d_X) \to (Y, d_Y)$ be a function between metric spaces. Define the *graph of* f to be

$$\Gamma_f := \{(x, f(x)) : x \in X\} \subseteq X \times Y.$$

Equip $X \times Y$ with the product metric from HW3 Q5,

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2} \text{ for all } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Prove that if f is continuous then its graph Γ_f is a closed subset of $(X \times Y, d_{X\times Y})$.

Remark: The converse also holds, so that f is continuous if and only if its graph is closed.

4. Record a video in which you discuss the following capstone result:

The continuous image of a compact set is compact. As a corollary, we have the extreme value theorem: any real-valued continuous function with compact domain is bounded and attains its bounds.

In your own words, you should explain the meaning of this result, outline the proof in reasonable but not necessarily total detail, and provide an example as well as a non-example. The video should include mathematical formulas as my lectures do, which you can write out by hand or in slides. Please keep the video under six minutes. **See *Canvas Announcements* or *Piazza* for the instructions on submitting your video, which must be done separately to the submission of this homework.**

5. Recall from HW2 Q4 how we define the distance between the subset of a metric space and a point in the metric space. If (M, d) is a metric space, $A \subseteq M$ nonempty and $x \in M$, then

$$d(x, A) := \inf_{a\in A} d(x, a).$$

(a) Show that if A is compact, then this infimum is attained, i.e., there exists $a^* \in A$ such that $d(x, a^*) = d(x, A)$.

(b) Give an example where A is not compact and the infimum is not attained.

6. [**Extra Credit**] Consider the great circle C on the surface of Earth passing through your favorite point in Austin, Texas and the North Pole. Prove that at any instant in time there must exist two *antipodal* (or *diametrically opposite*) points $p, p' \in C$ at which the surface temperature exactly matches. Please assume the surface of Earth is a perfect sphere so that C is a perfect circle, and that temperature is a continuous real-valued function of position.

Real Analysis: *Homework 7*

Due: 23:59 CDT, April 13, 2021

1. In this exercise, we gain experience in applying the fact the continuous image of a connected set is connected and in applying its corollary, the intermediate value theorem.

   (a) Consider a polynomial function $f : \mathbb{R} \to \mathbb{R}$ of odd degree. This means $f(x) = c_d x^d + c_{d-1}x^{d-1} + \ldots + c_0$ for $x \in \mathbb{R}$ and some fixed coefficients $c_d, \ldots, c_0 \in \mathbb{R}$ with $c_d \neq 0$ and d an odd positive integer. Prove that f must have a *real root*, i.e., there exists $r \in \mathbb{R}$ such that $f(r) = 0$. *Hint:* What can you say about $f(x)$ when x is a very large positive number and when x is a very large negative number? Please prove it. Now assuming HW6 Q2, apply the intermediate value theorem to conclude.

   (b) Recall HW2 Q1. Back then, one part of our solution was to construct a bijection from $(0, 1]$ to $(0, 1)$. Prove that there does *not* exist a *continuous* bijection $g : (0, 1] \to (0, 1)$. *Hint:* Assume for a contradiction that there does exist a continuous bijection $g : (0, 1] \to (0, 1)$. What can you say about $g\left((0, 1)\right)$?

2. (a) Directly using the $\varepsilon - \delta$ definition of the limit of a function, prove the *Squeeze Theorem*:

   Let $I \subseteq \mathbb{R}$ be an interval having $a \in \mathbb{R}$ as a limit point. Let $f, u, \ell : I \setminus \{a\} \to \mathbb{R}$. Assume
   - *$\ell(x) \leq f(x) \leq u(x)$ for all $x \in I \setminus \{a\}$, and*
   - *$\lim_{x\to a} \ell(x) = \lim_{x\to a} u(x) = L$ for some $L \in \mathbb{R}$.*

   Then $\lim_{x\to a} f(x)$ exists and equals L as well.

   (b) Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

   $$F(x) = \begin{cases} x + x^2 \sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

   Show that F is differentiable everywhere and determine $F'(x)$ for each $x \in \mathbb{R}$. *Hint:* To show F is differentiable at 0, use the Squeeze Theorem (part (a)).

   (c) For F as in part (b), show that while $F'(0) > 0$ there does *not* exist $\delta > 0$ such that F is monotonically increasing on $(-\delta, \delta)$. *Hint:* Since F' exists, show from the limit definition of a derivative that if F were monotonically increasing on $(-\delta, \delta)$ then we would have $F'(x) \geq 0$ for all $x \in (-\delta, \delta)$. Now consider the expression for $F'(x)$ from part (b).

(d) The behavior in part (c) is possible only because F' is not continuous at 0. Please prove that F' is not continuous at 0 by exhibiting a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ with $x_n \to 0$ and $F'(x_n) \nrightarrow F'(0)$ as $n \to \infty$. Nevertheless, F' does "assume intermediate values" (in agreement with Theorem 5.12 of Rudin). Please directly verify that for all $\lambda \in \mathbb{R}$ with $F'(0) < \lambda < F'(\frac{1}{\pi})$ there exists $x \in (0, \frac{1}{\pi})$ such that $F'(x) = \lambda$.

3. Suppose $f'(x) > 0$ for all $x \in (a, b)$. Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \text{ for } x \in (a, b).$$

Hint: Like in class, use the mean value theorem to show that f is strictly increasing. So f is a bijection onto its image, and g *exists*. Next, you can show g *is continuous* at each point of its domain by considering a suitable restriction of f and remembering the fact a continuous bijection out of a compact interval has a continuous inverse (Lecture 20). Finally, you can prove g *is differentiable* by directly inspecting the limit definition of a derivative applied to g.

4. Let $f : [a, b] \to \mathbb{R}$. We say that f is *Lipschitz continuous* if there exists a constant $C \in \mathbb{R}$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in [a, b]$.

   (a) Prove that if f is Lipschitz continuous then f is continuous.

   (b) Prove that if f is *continuously differentiable* (which is to say f' exists everywhere on $[a, b]$ and is a continuous function) then f is Lipschitz continuous. *Hint:* Mean value theorem.

   (c) Suppose on the other hand that $f : [a, b] \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in [a, b]$. Prove that f is constant.

5. Suppose that f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2} = f''(x).$$

Hint: Use L'Hôspital's rule (just once) and then use the limit definition of a derivative.

Real Analysis: *Homework 8*

Due: 23:59 CDT, April 30, 2021

`The first two exercises are based on Taylor's theorem (Lecture 24). The second is for extra credit. The last three exercises are about Riemann integrability.`

1. Let $f(x) = \sqrt{x}$. Express $f(1+h)$ as a quadratic in h plus a remainder term involving h^3. By taking $h = -0.02$, find an approximate value for $\sqrt{2}$ and prove it is accurate to seven digits. *Hint*: Notice $\sqrt{0.98} = 0.7\sqrt{2}$.

2. [**This entire exercise is for extra credit.**] Suppose $f : [a, b] \to \mathbb{R}$ is twice differentiable, $f(a) < 0$, $f(b) > 0$, there exists $\delta > 0$ such that $f'(x) \geq \delta$ for all $x \in [a, b]$ and there exists $M \geq 0$ such that $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. So f is strictly increasing (by $f' > 0$) and convex (by $f'' \geq 0$). Since f is continuous (being differentiable), strictly increasing and $f(a) < 0$ and $f(b) > 0$, please note there exists a unique $\xi \in (a, b)$ such that $f(\xi) = 0$. In this bonus exercise, we are going to develop *Newton's method*, which is an important numerical method for computing ξ. One sometimes describes Newton's method as a *root-finding* method.

   (a) By trying various values of f, suppose we find $x_1 \in (a, b)$ such that $f(\xi) > 0$. Then notice $x_1 \in (\xi, b)$ since f is strictly increasing and $f(\xi) = 0$ by definition of ξ. Let us now define a sequence of real numbers $(x_n)_{n=1}^{\infty}$ by the recursion
   $$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \quad (n \in \mathbb{N}).$$
   Please interpret this formula geometrically in terms of a tangent to the graph of f. *Remark*: Generating such a sequence is what we mean by running Newton's method. We call each iteration (passing from x_n to x_{n+1}) a *Newton step*.

   (b) Using the Mean Value Theorem, prove $\xi \leq x_{n+1} < x_n$ $(n \in \mathbb{N})$. Also show
   $$\lim_{n\to\infty} x_n = \xi.$$

   (c) Using Taylor's theorem, show that there exists $t_n \in (\xi, x_n)$ $(n \in \mathbb{N})$ such that
   $$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2.$$

   (d) Set $A := M/(2\delta)$. Using part (c) and the assumed bound on f'', deduce
   $$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n} \ (n \in \mathbb{N}).$$

Remark: Notice the exponent is 2^n, which grows rapidly with n. So, if the bracketed quantity $A(x_1 - \xi)$ is strictly less than 1 (this depends on the initialization x_1), then the bound on the RHS above decays rapidly to 0 indeed. The number of *correct digits* of x_n (as compared to ξ) *roughly doubles* every time we perform a constant number of Newton steps. In numerical analysis, one refers to this behavior as *quadratic convergence*.

(e) Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^{1/3}$. Using computer software or a calculator, perform Newton's method (the procedure in part (a)). What happens? Please reconcile.

3. (a) Let $f : [0,1] \to \mathbb{R}$ be the *indicator function* of $\mathbb{Q} \cap [0,1]$. This means, for $x \in [0,1]$,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is not Riemann integrable. *Hint*: What is $\mathcal{U}(f)$? What is $\mathcal{L}(f)$?

(b) Recall HW6 Q1(c). Consider the function defined there restricted to $[0,1]$. That is, let $g : [0,1] \to \mathbb{R}$ be given by

$$g(x) = \begin{cases} 1/q & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms with } p \in \mathbb{Z}, q \in \mathbb{Z}_{>0}; \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that g is Riemann integrable and $\int_0^1 g = 0$. *Hint*: Try to verify the $\varepsilon - \mathcal{P}$ characterization of integrability.

4. Let $f : [a,b] \to \mathbb{R}$ be increasing throughout $[a,b]$ (i.e., $f(x) \leq f(y)$ whenever $a \leq x \leq y \leq b$). Prove that f is integrable on $[a,b]$. *Hint*: Verify the $\varepsilon - \mathcal{P}$ characterization of integrability by considering a partition of equispaced breakpoints.

5. Recall the "polished" definition I presented in lecture of the Riemann integral: a bounded function $f : [a,b] \to \mathbb{R}$ is *integrable* with $\int_a^b f = A$ if its upper integral and lower integral both equal A, that is, $\mathcal{U}(f) = A = \mathcal{L}(f)$. Actually, Riemann originally defined his integral differently, in a way corresponding to the *Riemann sums* you may have seen in a calculus class:

Riemann's Original Definition: *A bounded function* $f : [a,b] \to \mathbb{R}$ *is* integrable *with* $\int_a^b f = A$ *if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any partition* $\mathcal{P} = \{x_0, \ldots, x_n\}$ *of* $[a,b]$ *with* $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ *and any* sample points $c_k \in [x_{k-1}, x_k]$ *(each* k*) such that* $x_k - x_{k-1} < \delta$ *(each* k*), it holds that*

$$\Big| \sum_{k=1}^{n} (x_k - x_{k-1}) f(c_k) - A \Big| < \varepsilon.$$

Prove if f satisfies Riemann's original definition, then f satisfies the definition from lectures. *Remark*: Actually, the converse holds too (proof omitted). So Riemann's original definition is equivalent to the one from lectures. In your opinion, is the construction I gave easier?

Real Analysis: *Homework 9*

Due: 23:59 CDT, May 9, 2021

`The first two exercises are based on the Fundamental Theorem of Calculus. The last three exercises are about uniform convergence.`

1. Record a video in which you discuss the following capstone result:

   The Fundamental Theorem of Calculus (both parts).

   In your own words, you should explain the meaning / relevance of this result, say a few words about its proof and what the ingredients are, and provide an example for each part of FTC. The video should include mathematical formulas as my lectures do, which you can write out by hand or in slides. Please keep the video under six minutes. **See *Canvas Announcements* or *Piazza* for the instructions on submitting your video, which must be done separately to the submission of this homework.**

2. Let $g : [a, b] \to \mathbb{R}$ be differentiable and assume $g' : [a, b] \to \mathbb{R}$ is continuous. Let $f : [c, d] \to \mathbb{R}$, be continuous and assume the range of g is contained in $[c, d]$, so that the composition $f \circ g$ is properly defined.

   (a) Why is f the derivative of some function? How about $(f \circ g)g'$?

   (b) Prove the change-of-variable formula

   $$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt.$$

3. Let $f_n, g_n : E \to \mathbb{R}$ be functions on a set E. Assume $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ converge uniformly.

   (a) Prove that $(f_n + g_n)_{n=1}^{\infty}$ converges uniformly.

   (b) In addition, assume f_n and g_n (each n) are bounded functions. Prove that $(f_n g_n)_{n=1}^{\infty}$ converges uniformly.

4. Prove that the series

   $$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

   converges uniformly in every bounded interval of $\mathbb{R}$, but does not converge absolutely for any value of x.

5. Let $\{x\} \in [0,1)$ be the *fractional part* of $x \in \mathbb{R}$. This means $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor \in \mathbb{Z}$ is the *floor* of x, i.e., the greatest integer less than or equal to x. Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2} \quad (x \text{ real}).$$

Find all discontinuities of f, and show that they form a countable dense subset of $\mathbb{R}$. Nonetheless, show that f is Riemann-integrable on every bounded interval.