THREE-MANIFOLD GEOMETRY

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INTRODUCTION

We’re going to be talking about orientable manifolds throughout; the case of non-orientable manifolds is not much more complicated, and can be read about, for example, in Scott [6]. Note as well that in dimension three, topological manifolds admit unique (up to diffeomorphism) smooth structures, hence classification up to homeomorphism is the same thing as classification up to diffeomorphism. Furthermore, we will always be considering connected manifolds.

To break my own rule: in the case of closed surfaces, genus entirely determines geometry. All closed (compact, boundaryless) surfaces can be obtained as a quotient of one of $S^2$, $E^2$, or $H^2$ by a subgroup of the isometry group. This choice of universal cover is unique so long as one requires that the metric obtained on the quotient be geodesically complete.

What does this mean? We know what all the closed surfaces are, and the above classification aligns with classification by genus. The genus zero closed surfaces, $S^2$ and $RP^2$, are obtained as quotients of $S^2$, and by the above paragraph we know that it is impossible to put a geodesically complete Riemannian metric on $S^2$ or $RP^2$ which is not, at each point, exactly equal to the metric at each point in the associated equivalence class of points in $S^2$. The genus one closed surfaces, $T^2$ and the Klein bottle, similarly are quotients of $E^2$. All other closed surfaces are quotients of $H^2$.

In dimension three, we have Poincaré’s Question:

**Theorem 1.** Is any closed 3-manifold with trivial fundamental group homeomorphic to $S^3$?

I call it a question, since Poincaré did not phrase it as a conjecture. We also have it’s more general parent, the Geometrization Theorem:

**Theorem 2.** (Perelman, etc.) The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures.

To simplify matters, we’ll talk about closed, oriented manifolds throughout. The purpose of this talk is to understand the statement of the above theorem and to gain an appreciation for the breadth of work that went into its proof, which shaped the study of three-manifolds in the twentieth century.

**What is “Geometry”?**

What we want is an analog of Euclidean plane geometry, relating lines, angles, and distances, with notions of rotation, reflection, and parallelism. A powerful way of making this analogy is to study structure-preserving maps. One way to codify the idea of “structure” is to use a metric.
**Definition** A Riemannian metric $g$ on a manifold $M$ is a smooth choice of inner product on $T_pM$ for all $p \in M$.

In the following, we will often use “metric” in the place of “Riemannian metric”; the cases where “metric” is meant in the sense of distance function will be obvious.

Metrics allow us to build the analogy we were looking for. A metric gives us a way to define a metric space structure on a manifold. If $\gamma : [a, b] \to M$ is piecewise smooth, then its length is given by $\int_a^b g(\gamma'(t), \gamma'(t)) \, dt$; the distance from $p$ to $q$ is then the infimum over all such curves of their lengths. We can then make a generalization of the idea of a Euclidean line with the notion of a geodesic, or a locally length-minimizing curve (there is an equivalent notion as a constant-speed curve).

One can go from a metric to a group of structure-preserving maps simply by considering the group of diffeomorphisms of $M$ which preserve the metric in the following sense.

**Definition** Let $(M, g)$ be a Riemannian manifold, and let $f : M \to M$ be a diffeomorphism. $f$ is an isometry of $(M, g)$ if for all $p$ in $M$, $f^*g = g$. That is, for all $X_p, Y_p \in T_pM$,

$$g_{f(p)}(df_p(X_p), df_p(Y_p)) = g_p(X_p, Y_p)$$

By the discussion about lengths and angles, one can translate between the idea of an isometry as a metric-preserving transformation and as a line/circle and angle preserving transformation.

Since diffeomorphisms compose to other diffeomorphisms, the set of isometries of a manifold is a group. However, given a group, it is not always clear when there exists a manifold for it to act as an isometry group of.

**Aside: The Basic Model Geometries**

Before we make further abstractions, let’s consider the three major geometries.

Euclidean geometry, $\mathbb{E}^n$, is the most familiar. You can think of $\mathbb{E}^n$ as a homeomorphic copy of $\mathbb{R}^n$ with the standard Euclidean inner product as the metric. Lines and angles are standard lines and angles. A pair of points in $\mathbb{E}^n$ shares a unique geodesic. Geodesics intersect in either one point or zero points. Given a line $L$ and a point $P$ off $L$, there is precisely one line $L'$ which contains $P$ and does not intersect $L$. The angles of a triangle in $\mathbb{E}^2$ (or in any totally geodesic 2-submanifold) always add to $\pi$.

Spherical geometry, $S^n$, is the next most familiar. $S^n$ is a homeomorphic copy of $\mathbb{S}^n$, with the “round” metric induced by the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$. Geodesics are great circles, and you can measure the angles simply by placing a Euclidean protractor tangent to the sphere at the point of interest. A pair of points in $S^n$ shares a geodesic, but if they are antipodal, it is not unique. All geodesics intersect in two points. The angles of a triangle in $S^2$ (or in any totally geodesic 2-submanifold) always add to more than $\pi$, and their sum determines its area.

Hyperbolic geometry, $\mathbb{H}^n$, is the least familiar (in part, because it is impossible to embed a smooth hyperbolic surface in Euclidean space) and yet the most general (most 3-manifolds are hyperbolic, in a sense to be explained below). It is easiest to understand by way of a model in $\mathbb{E}^n$. There are several of these: the one we will see is the Poincaré ball model. Take the unit ball in $\mathbb{E}^n$, and declare all circles orthogonal to the boundary (including diameters) to be geodesics. You can measure angles as in the Euclidean space in which our model is embedded (thought it takes some work to see this). A pair of points in $\mathbb{H}^2$ shares a unique geodesic. Geodesics intersect in either one point or zero points. Given a geodesic $L$ and a point $P$ off $L$, there are infinitely many many geodesics $L'$
which contain \( P \) and do not intersect \( L \). The angles of a triangle in \( \mathbb{H}^2 \) (or in any totally geodesic 2-submanifold) always add to less than \( \pi \).

**Back to “Geometry”**

How can \( \mathbb{E}^n \), \( \mathbb{S}^n \), and \( \mathbb{H}^n \) help us understand 3-manifolds? In a sense, their symmetries are all the symmetries which could possibly exist, at least in dimension three. We are going to make precise that “sense.” The idea is that a geometry is a space paired with a distinguished set of isometries – though we don’t know they are isometries yet. Much of this section is from Thurston [8].

**Definition** A model geometry \((G,X)\) is a (smooth, though it doesn’t matter in dimension 3) manifold \( X \) together with a Lie group \( G \) of diffeomorphisms of \( X \) such that the following hold:

(a): \( X \) is connected and simply connected.

(b): \( G \) acts transitively on \( X \), with compact point stabilizers.

(c): \( G \) is not contained in any larger group of diffeomorphisms of \( X \) with compact point stabilizers.

(d): There exists at least one compact manifold modeled on \((G,X)\).

There are several things to mention at this point. Some are definitions, and some are remarks about why we care about the above mentioned properties of a group of diffeomorphisms.

**Definition** A manifold \( M \) is a \((G,X)\)-manifold or modeled on \((G,X)\) if it admits an atlas of charts \( \{U_i, \varphi : U_i \to X\} \) where the \( \varphi \) are homeomorphisms onto their images such that if \( U_i \cap U_j \neq \emptyset \) then the restriction of \( \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j) \) to connected components agrees with the restriction of some element of \( G \). The atlas is called a \((G,X)\) structure.

We need (d) for if we require only (a)-(c) it is possible to create model geometries which do not model any compact manifolds. (Modeling noncompact manifolds is generally not very hard, and not unique.)

Earlier, I mentioned that metrics determine geometries. It is possible to show ([8]) that \( X \) admits a \( G \)-invariant Riemannian metric if and only if \( G \) acts transitively on \( X \) with compact point stabilizers. With a little more work (also [8]) it is possible to show that for any such \((G,X)\), the induced (via pullback through the charts and perhaps a partition of unity) metric on a closed \((G,X)\)-manifold is always geodesically complete (geodesics exist for all time – compare with geodesics on \( \mathbb{R}^2 \) with the metric obtained via pullback of the standard metric on the the Riemann sphere, for an example of a metric without geodesic completeness). This is the significance of (b).

(Another reason for the transitivity requirement in (b): we want all points in the geometry to “look” the same, in terms of either the metric or – equivalently – the isometry group.)

We require (c) essentially for simplicity; given an isometry class of metrics, we’d like them all to have the same isometry group.

**Geometry and Topology**

Let’s make explicit the connections between these viewpoints of geometry: the metric viewpoint, and the isometry group viewpoint. Notice that I haven’t yet talked about (a); its role will become clear in this section.
Given a \((G, X)\)-manifold \(M\), one can define a map \(D : \tilde{M} \to X\) called the developing map \(D : \tilde{M} \to X\). One can see [8] for a thorough explanation of this map; the idea is to think of \(\tilde{M}\) as the space of homotopy classes of based paths in \(M\), and to imagine the developing map as “unrolling” a small neighborhood of \([\alpha] \in \tilde{M}\) along \(\alpha\).

For example, when developing a torus obtained as a quotient of \(\mathbb{E}^n\) by a lattice, one gets all of Euclidean space. In this case the torus is \(M\), and Euclidean space is both \(\tilde{M}\) and \(X\), and the developing map serves as a homeomorphism between them. When developing a torus obtained from a quadrilateral in \(\mathbb{R}^n\) which is not a parallelogram, the developing map cannot cover all of \(\mathbb{R}^n\) and generally omits a point.

**Definition** If \(D : \tilde{M} \to X\) is a covering map, \(M\) is a complete \((G, X)\)-manifold.

This brings us back to (a) in the definition of a model geometry. We now see why we’d like \(X\) to be simply connected; a covering map is a homeomorphism when \(X\) is simply connected, and we’d like \(\tilde{M} \cong X\).

The terminology “complete” may make you wonder if developing completeness has anything to do with geodesic completeness; it can be shown ([8]) that they are equivalent (that geodesic completeness is equivalent to metric space completeness with the metric induced by the Riemannian metric is the well-known Hopf-Rinow theorem).

The final component is the idea of holonomy. Given an element \([\gamma] \in \pi_1(M)\), developing along \([\gamma] \in \tilde{M}\) brings us back to the basepoint in \(M\), so one can compare an original chart about the basepoint to the new chart obtained by developing. This “comparison” is going to be some element of \(G\), that is, some diffeomorphism of the model space. The map from \(\pi_1(M)\) to \(G\) given in this way is called the holonomy of \(M\). It is a homomorphism, and when \(M\) is complete and \(X\) simply connected one can show that it is injective.

Using the above paragraph, we can now relate the topology of \(M\) (\(\pi_1\)) to its geometry (models it admits). On the one hand, we have:

**Proposition 1.** (paraphrased from Thurston) For \(G\) a group of diffeomorphisms of a simply connected space \(X\), any complete \((G, X)\)-manifold may be reconstructed as \(X/\Gamma\) where \(\Gamma\) is the image of \(\pi_1(M)\) under the holonomy.

While on the other hand, we have:

**Proposition 2.** (paraphrased from Thurston) If \(G\) is a Lie group acting transitively with compact point stabilizers on a simply connected manifold \(X\), and \(M\) a differentiable manifold, then \((G, X)\)-structures on \(M\) are in one-to-one correspondence with conjugacy classes of discrete subgroups of \(G\) which are isomorphic to \(\pi_1(M)\) and act freely on \(X\) with quotient \(M\).

If \(M\) is closed, recall that from our discussion of (b) in the definition of a model geometry, one can simply look at all \((G, X)\) structures on \(M\), since they are necessarily complete.

The condition of a free action (no element of \(G\) fixes all of \(X\) besides the identity) is one of the requirements for the quotient to be a manifold. Otherwise, we get orbifolds.

Note for the interested reader: one should in fact be considering only real analytic actions in much of the above section. However, this is not an issue, since by a theorem of Whitney ([9]) one can always “stiffen” a smooth structure into a real analytic one. I decided to omit the discussion of analyticity for simplicity.
In what sense are $E^n$, $S^n$, and $H^n$ “enough” to describe all 3-manifold geometry?

**Theorem 3. (Thurston)** There are eight 3-dimensional model geometries $(G, X)$: the usual $E^3$, $S^3$, $H^3$, the products $S^2 \times E^1$ and $H^2 \times E^1$, and the twisted products $Nil$ and $\widetilde{SL_2}(\mathbb{R})$ (both $E^1$ bundles, over $E^2$ and $H^2$, respectively) and $Solv$ (a $E^2$ bundle over $E^1$).

Topologically, many of these are just $\mathbb{R}^3$; but their metrics and isometry groups are all very different. For an in-depth discussion of all the geometries, see either Thurston ([7],[8]) or Scott ([6]). The product geometries are the edge cases; in fact, all non-hyperbolic 3-manifolds are classified (see [6]).

One can show the following (which also shows that none of the geometries are equivalent):

**Theorem 4. (Scott)** If $M$ is a closed 3-manifold which admits a $(G, X)$-structure for one of the above geometries, then the geometry involved is unique.

**Decomposition**

References for much of this section are Hatcher ([1]) and Scott ([6]), as well as a lecture given by Scott at UC Davis on 3-manifolds (google it).

At this point, one might ask: does every 3-manifold admit a geometric structure? Not even close; take a connected sum of manifolds modeled on two different geometries. The next natural thing to do is to consider manifolds which cannot be broken down into simpler manifolds via the operation of connected sum. We will be considering this in the opposite order that it usually is presented.

**Definition** We say $M^3$ is the connected sum of $M_1$ and $M_2$ (denoted $M = M_1 \# M_2$) if there is an embedded $S^2$ disjoint from $\partial M$ which separates $M$ into two components. Then $M_1$ and $M_2$ are obtained from those components by gluing a $B^3$, and are closed if $M$ was.

**Definition** We say $M^3$ is prime if every connected sum decomposition of $M$ is of the form $M \# S^3$.

Note that being prime is equivalent to the statement “every separating $S^2$ in $M$ bounds a $B^3$.”

Kneser ([3]) showed that any compact 3-manifold can be expressed as a connected sum of finitely many prime manifolds; Milnor ([4]) showed that if the original manifold is orientable, the factors are unique up to insertion/deletion of $S^3$s.

Now is it the case that every prime 3-manifold admits a geometric structure? No: examples are given by a class of manifolds called Seifert fiber spaces. Take two such manifolds with boundary and glue them along their boundary with a non-identity mapping (for example, one could take $D^2 \times S^1$).

To further simplify the situation, let’s consider what we did when splitting up into primes: we cut along embedded separating $S^2$s which did not bound $B^3$s.

**Definition** $M^3$ is irreducible if every embedded $S^2$ bounds a $B^3$.

It is not to hard to show (see Hatcher [1]) that the only prime 3-manifold which is not irreducible is $S^2 \times S^1$ (it is not very hard at al to see that $S^2 \times S^1$ is prime and reducible; try it). The rest are irreducible. How should we simplify these? The next simplest surface from $S^2$ is $T^2$, so let’s try that.
Definition $S^2$ embedded in $M^3$ is \textit{incompressible} if the map on $\pi_1$ induced by the embedding is injective.

The decomposition of manifolds due to Jaco and Shalen and Johannson has to do with splitting an irreducible, compact, orientable $M^3$ along disjoint incompressible tori. It is the resulting pieces into which $M$ splits which ought to admit geometric structures. This splitting, if taken to be minimal, is unique up to isotopy in $M$.

It turns out that there are two kinds of manifolds into which $M$ splits when cut along tori: Seifert fiber spaces and “others.” What was so hard about the Geometrization Conjecture was showing that these other components all had to be hyperbolic. In the case of Haken manifolds (in the compact, orientable, irreducible case, this means it contains an orientable incompressible surface), it was possible to show this (and Thurston did), but no-one had any idea how many manifolds were Haken.

\textbf{Geometrization Conjecture Implies Poincaré Conjecture}

(Following Kapovich [2].) Let $M^3$ be closed and simply connected, with a connected sum decomposition $M_1 \# \cdots \# M_n$ into primes. It can be shown (using Van Kampen’s theorem; see the discussion in Hatcher [1]) that $\pi_1(M) = \pi_1(M_1) \ast \cdots \ast \pi_1(M_n)$, so $M_i$ is simply connected for all $i$. So $M_i$ can contain no incompressible tori, so each $M_i$ admits a geometric structure. It can be shown that if $\pi_1(N)$ is finite then $N$ is modeled on $S^3$ (see Scott [6]), and since $M_i$ is simply connected, $M_i = S^3$. So $M = S^3$.

\textbf{Resolution and Ricci Flow}

The solution to this conundrum turned out to be the idea of “flowing” a metric until it admits almost constant curvature, then cutting the manifold up based on the curvature on each piece. The curvature of a metric is outside the scope of this talk; I found Kapovich’s [2] exposition to be pretty elucidating.

\textbf{Resources}

Beyond what I have directly drawn from for this talk, there are plenty of other resources for the interested reader. In particular, see Milnor’s survey article [5] for resources on the history of 3-manifold classification – most of the items in my bibliography have extensive and elucidating bibliographies of their own – and the archive of Kleiner and Lott (at math.berkeley.edu/ lott/ricciflow/perelman.html) for resources on Perelman and Hamilton’s use of the Ricci flow.

\textbf{References}
