Large Cardinals 101

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The Cumulative Hierarchy

Definition

We construct the cumulative hierarchy by taking:

- \( V_0 = \emptyset \)
- \( V_{n+1} = \mathcal{P}(V_n) \)
- \( V_\lambda = \bigcup_{\alpha \in \lambda} V_\alpha \) for limit \( \lambda \)
- \( V = \bigcup_{\alpha \in \mathbb{ON}} V_\alpha \) (foundation)

Remark

Successor levels at infinite stages are not necessarily well-defined as there is no procedure for producing all subsets of a given level. Also, note that, by induction, \( \kappa \subset V_\kappa \) and \( V_{\kappa+1} \) is the first level such that \( \kappa \in V_{\kappa+1} \).
The Constructible Hierarchy

Definition
Gödel’s constructible universe is similarly constructed:

- $L_0 = \emptyset$
- $L_{n+1} = \text{first-order definable subsets of } L_n \text{ with parms from } L_n$
- $L_\lambda = \bigcup_{\alpha \in \lambda} L_\alpha$ for limit $\lambda$
- $L = \bigcup_{\alpha \in \omega} L_\alpha$

Remark
$V_n = L_n$ for $n \in \omega$ and so $V_\omega = L_\omega$. But, $|V_{\omega+1}| = 2^\omega$ while $|L_{\omega+1}| = \omega$. $L_{\omega+1}$ only contains the arithmetic (i.e., $\Sigma^0_n$ or $\Pi^0_n$) subsets and relations on $\omega$. The last of the hyperarithmetic subsets of $\omega$ show up in $L_{\omega_1}$, but are all in $V_{\omega+1}$. We also define $L[A]$ by allowing all f.o. definable subsets of $L_n$ using parm $A$ and parms in $L_n$ (but $A$ will not necessarily be an element of the universe). $L(A)$ is defined as the smallest constructible model containing the set $A$ (but Choice might not hold in this model).
Inner Models of ZFC

Definition
An inner model of ZF is a transitive class that contains all ordinals and satisfies the axioms of ZF.

Proposition
$L$ is the smallest inner model of ZF. Additionally, $L \vdash AC + GCH$.

Proof.
(Partial) If $M$ is an inner model, then $L^M$ (the class of all constructible sets in $M$) is $L$. Hence $L \subseteq M$.

Remark
$V = L$ is independent of ZFC. While not directly hypothesizing the existence of a large cardinal, it does limit what large cardinals can exist and, so, is a type of large cardinal axiom. However, note that direct large cardinal hypothesis tend to not decide $CH$ and $GCH$. 
We can calculate these, so no existence hypotheses are required (they exist whether $V = L$ or $V \neq L$):

- $\aleph_1, \aleph_2, \ldots$
- $c = 2^{\aleph_0}$ ($= \aleph_1$ if $V = L$)
- $\aleph$-fixed points: Take $\alpha_0 = \aleph_0$, $\alpha_{n+1} = \aleph_{\alpha_n}$, and $\alpha = \sup \{\alpha_n : n \in \omega\}$ (i.e., $\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}} \ldots$). Then $\alpha = \aleph_{\alpha}$.
- $\beth$-fixed points: Take $\beta_{n+1} = \beth_{\beta_n}$ and $\beta = \sup \{\beta_n : n \in \omega\}$. Then $\beta = \beth_\beta$. (if $V = L$, $\aleph_\alpha = \beth_\alpha$, i.e., GCH holds)
Types of Large Cardinal Characterizations

- Closure
- Combinatorial
- Filters/Ultrafilters
- Embeddings and Reflection Properties
- Indescribability
- Logic Compactness
Closure

Closure conditions ensure that if a certain cardinal (or ordinal) is below a given cardinal, then so are many others. The poster child is:

Example

$k$ is a strong limit cardinal if $\gamma < k$, then $2^\gamma < k$.

- Closure conditions can also take the form: There exists $k$ having some-large-cardinal-property with $k$ cardinals below it having the same-large-cardinal-property.
- These can also be viewed as fixed point existence hypothesis.
Combinatorial

These are generalizations of the pigeon hole principle and may be stated as properties of partitions or trees.

Definition
\(\kappa \rightarrow (\lambda)^\alpha_\beta\) if for every partition of the set \([\kappa]^\alpha\) into \(\beta\) pieces, there exists a subset \(H\) of \(\kappa\) of size \(\lambda\) such that \([H]^\alpha\) lies in a single partition.

Example
In this notation, the pigeon hole principle is written as \(\omega \rightarrow (\omega)_n^1\) for \(n \in \mathbb{N}\).

Example
A regular uncountable \(\kappa\) has the **tree property** if every tree of height \(\kappa\) whose levels have cardinality \(< \kappa\) must have a branch of cardinality \(\kappa\). (This is a generalization of Konig’s lemma)
Embeddings and Reflection

Definition

$N$ is an **elementary substructure** of $M$ if $M \models \varphi \iff N \models \varphi$ for every first order formula with parameters from $N$. A map $j : N \to M$ is an elementary embedding if it is an injective homomorphism and $j'' N$ is an elementary substructure of $M$. An elementary embedding is **nontrivial** if it is not the identity. The **critical point** of a nontrivial elementary embedding $j$ is the least ordinal $\kappa$ such that $j(\kappa) > \kappa$.

Fact

*(Reflection Principle)* $\text{ZFC} \models \forall n \in \mathbb{N} \forall \alpha \in \mathbb{ON} \exists \beta > \alpha$ such that $V_\beta \models \varphi \iff V \models \varphi$ for any $\varphi$ a first-order formula with fewer than $n$ quantifiers.

Large cardinal characterizations can assert stronger reflection properties. With regard to embeddings, “reflection” happens through the critical point of the embedding and this gives rise to large cardinals because, if something happens above the critical point, a whole lot of that something exists below the critical point.
Ultrafilters...

Definition
An ultrafilter on a nonempty set $S$ is a collection $U$ of subsets of $S$ such that:

1. $S \in U \land \emptyset \notin U$
2. $X \in U \land Y \in U \Rightarrow X \cap Y \in U$ (closed under pairwise intersections)
3. $X, Y \subset S \land X \in U \land X \subset Y \Rightarrow Y \in U$ (closed under supersets)
4. $\forall X \subset S (X \in U \lor S - X \in U)$ (maximal)

Definition
If $\exists X_0 \subset S$ such that $\forall X \in U \Rightarrow X_0 \subset X$, then $U$ is principal, otherwise it is non-principal.

Definition
$U$ is $\kappa$-complete if it is closed under intersection of less than $\kappa$ elements (where $\kappa$ is a regular cardinal).
...and Ultrapowers...

Let $\{\mathcal{A}_x : x \in U\}$ be a collection of models of ZFC and $U$ an ultrafilter and consider $\prod_{x \in U} A_x$.

**Definition**

$f =_U g$ iff $\{x \in S : f(x) = g(x)\} \in U$.

Now, let $A = \prod_{x \in U} A_x / =_U$ and interpret the model $\mathcal{A}$ with domain $A$ by:

- For a relation $R$,
  $R^{\mathcal{A}}([f_1], ..., [f_n]) \iff \{x \in S : R^{\mathcal{A}_x}(f_1(x), ..., f_n(x))\} \in U$

- For a function $F$, $F^{\mathcal{A}}([f_1], ..., [f_n]) = [f]$ where $f(x) = F^{\mathcal{A}_x}(f_1(x), ..., f_n(x))$ for all $x \in S$

- For a constant $c$, $c^{\mathcal{A}} = [f]$ where $f(x) = c^{\mathcal{A}_x}$ for all $x \in S$

**Definition**

If each $\mathcal{A}_x = \mathcal{A}$ in the above construction (i.e., they are all the same model), then the resulting model $\mathcal{A} = \text{Ult}_U \mathcal{A}$ is called the ultrapower of $\mathcal{A}$. 
...and How Ultrapowers Yield Embeddings

Theorem
(Łoś) If $U$ is an ultrafilter on $S$ and $\mathcal{A} = \text{Ult}_U A$, then for a formula $\varphi$ and a sentence $\psi$:

$\mathcal{A} \models \varphi([f_1], \ldots, [f_2]) \iff \{ x \in S : \mathcal{A} \models \varphi[f_1(x), \ldots, f_n(x)] \} \in U$

$\mathcal{A} \models \psi \iff \{ x \in S : \mathcal{A} \models \psi \} \in U$

That is, $\mathcal{A}$ is elementarily equivalent to $\mathcal{A}$. Even more, it can be embedded in its ultrapower $\mathcal{A}$:

Definition
The ultrapower embedding $j : \mathcal{A} \rightarrow \text{Ult}_U A$ is defined by $a \mapsto [c_a]$ where $c_a(x) = a$ for every $x \in S$. 
Indescribability

It may not be possible to differentiate (from below) between a large cardinal and a smaller cardinal by a formula of a given complexity. When this happens, it means there are LOTS of these similar smaller cardinals, pumping up the size of the larger cardinal.

Definition
A formula $\varphi$ is $\Pi^m_n$ if it is a formula of order $m+1$ of the form $\forall X\exists Y \ldots \psi$ where there are $n$ quantifiers and each $X, Y \ldots$ are $(m+1)$-th order variables and $\psi$ is such that all quantified variables are of order at most $m$.

Definition
A cardinal $\kappa$ is $\Pi^m_n$-indescribable if whenever $U \subset V_\kappa$ and $\varphi$ is a $\Pi^m_n$ sentence such that $(V_\kappa, \in, U) \models \varphi$, then for some $\alpha < \kappa$, $(V_\alpha, \in, U \cap V_\alpha) \models \varphi$.

That is, whenever $\varphi$ is $\Pi^m_n$, some smaller initial segment of the universe $V_\kappa$ (along with the restriction of the predicate) believes the formula to be true.
Logic Compactness

- Denote by $L_{\omega,\omega}$ the first-order logic allowing finite conjunctions/disjunctions of formulas on a countable language. A theory is defined as a set of sentences of the language. It is provable that, if each finite subset of the theory has a model (a.k.a. is finitely satisfiable), then the entire theory has a model. This is called the Compactness Theorem.

- Logical characterizations of large cardinals make some type of assertion of compactness for cardinals beyond $\omega$. 
Before We Start, Some More Definitions...

Definition
A set $X$ is unbounded in $\beta$ if $\forall \alpha \in \beta \exists x \in X (\alpha \leq x)$.

And now a very important definition:

Definition
The cofinality of $\beta$, $cf(\beta)$, is the least $\alpha$ such that there exists $f : \alpha \to \beta$ such that $\text{ran}(f)$ is unbounded in $\beta$. We say $f$ maps $\alpha$ cofinally into $\beta$.

- $cf(\beta) \leq \beta$ since $id : \beta \to \beta$.
- $\beta = \alpha + 1 \Rightarrow cf(\beta) = 1$
- From any cofinal map, we can define a strictly increasing cofinal map.
- Strictly increasing cofinal maps preserve cofinality and hence $cf(cf(\beta)) = cf(\beta)$. 
Definition
\( \alpha \) is **regular** if \( \alpha \) is a limit ordinal and \( cf (\alpha) = \alpha \). Otherwise, it is singular.

- \( \omega \) is regular.
- (AC) \( \kappa^+ \) is regular.
- \( cf (\omega \omega) = \omega \) and \( cf (\omega \lambda) = \lambda \) for any limit ordinal \( \lambda \).

Remark
If \( \omega \lambda \) is a regular limit cardinal, then \( \omega \lambda = \lambda \) (same argument as \( \kappa^+ \) regular). But the converse is not true. Take \( \alpha_0 = \omega, \alpha_{n+1} = \omega \alpha_n \), and \( \alpha = \sup \{ \alpha_n : n \in \omega \} \) (i.e., \( \omega, \omega \omega, \omega \omega \omega \ldots \)). Then \( \alpha = \omega_\alpha \) even though \( cf (\alpha) = \omega \). Note that this construction can be used to define a fixed point above any given ordinal/cardinal.
Inaccessibles

Definition
\(\kappa\) is strongly inaccessible (or just inaccessible) if it’s uncountable, regular, and strong limit (i.e., \(\forall \alpha < \kappa \ (2^\alpha < \kappa)\)).

Proposition
\(V_\kappa \models ZFC\)

Proof.
Extensionality: True relativized to any transitive class. Foundation: True in \(V_\kappa\). Pairing: If \(a, b \in V_\kappa\), then \(\exists \alpha < \kappa\) with \(a, b \in V_\alpha\). This implies \(\{a, b\} \in V_{\alpha+1} \subset V_\kappa\). Union: If \(a \in V_\kappa\), then \(a \in V_\alpha\) for some \(\alpha\). Since \(V_\alpha\) is transitive, \(a \subset V_\alpha\) so that \(\forall b \in a \ (b \in V_\alpha)\). This implies that \(\cup a \in V_{\alpha+1} \subset V_\kappa\). Power Set: If \(a \in V_\kappa\), then \(a \in V_\alpha\) for some \(\alpha\) and \(a \subset V_\alpha\). If \(b \subset a\), then \(b \in V_{\alpha+1}\) due to transitivity. This implies \(\mathcal{P}(a) \in V_{\alpha+2}\). Infinity: \(\omega \in V_\kappa\). Separation: Let \(X \in V_\kappa\) and \(\varphi(x, p)\) be a formula with parameter \(p \in V_\kappa\). Then \(X, p \in V_\alpha\) for some \(\alpha\) and \(X \subset V_\alpha\) so that \(\forall a \in X\) such that \(\varphi(a, p)\), \(a \in V_\alpha\). Hence \(\{x \in X : \varphi(x, p)\} \in V_{\alpha+1} \subset V_\kappa\). Replacement: Let \(A \in V_\kappa\) and \(\varphi\) be a formula such that \(\forall a \in A \exists! b \in V_\kappa \ (V_\kappa \models \varphi(a, b))\). \(|A| < \kappa\) as is \(B = \{b : \exists a \in A \ (V_\kappa \models \varphi(a, b))\}\). \(\kappa\) regular implies \(\forall b \in B \ (rk(b) < \kappa)\) which, in turn, implies \(B \subset V_\alpha\) for some \(\alpha \leq \kappa\). Hence \(B \in V_{\alpha+1} \subset V_\kappa\). Choice: Let \(X \in V_\kappa\) and let \(A \subseteq X\). Then \(X \in V_\alpha\) for some \(\alpha < \kappa\); \(A \subseteq X \subseteq V_\alpha\). Assume \(A\) contains an infinite descending chain \(\{a_i\}\). Then consider \(rk(a_i)\).
More on Inaccessibles

Proposition

If consistent, $ZFC \nvdash \exists \kappa$ inaccessible.

Proof.

If $\kappa$ is inaccessible, $V_\kappa \models ZFC$. Let $\kappa$ be the least inaccessible. Then $V_\kappa \models ZFC + "$ there is no inaccessible"."
Definition
A set $C \subset \kappa$ is **club** (closed, unbounded) in $\kappa$ if it is unbounded in $\kappa$ and contains all its limit points less than $\kappa$.

Remark
For $C, D$ club on $\kappa$, $C \cap D$ is club on $\kappa$. (construct two interlacing unbounded sequences, then their limits must be equal and in both clubs).

Definition
A set $S \subset \kappa$ is **stationary** if $\forall C$ club on $\kappa$, $S \cap C \neq \emptyset$.

- If $\kappa$ is the least inaccessible, then all strong limit cards below it are singular.
- If $\kappa$ is the $\alpha$-th inaccessible with $\alpha < \kappa$, then the set of regular cardinals below $\kappa$ is nonstationary.
Definition
An inaccessible cardinal $\kappa$ is **Mahlo** if the set of inaccessible cardinals below $\kappa$ is stationary.

Proposition
$\kappa$ Mahlo implies that $\kappa$ is the $\kappa$th inaccessible.

Proof.
If the set of all inaccessibles below a $\kappa$ is stationary, then $\kappa$ is a limit of inaccessible cardinals. Since $\kappa$ is regular, $cf(\kappa) = \kappa$ so that $\kappa$ must be the $\kappa$-th inaccessible.

Proposition
$CON(ZFC) \n \exists \kappa \ Mahlo$.

Proof.
Let $\kappa$ be the least Mahlo cardinal. Since it is inaccessible so that $V_\kappa \models ZFC$. However, $V_\kappa \models ZFC + \neg Mahlo$. 


Weakly Compact

Definition

A cardinal $\kappa$ is **weakly compact** if (assuming $\kappa^{<\kappa} = \kappa$):

1. It is uncountable and $\kappa \rightarrow (\kappa)^2_2$. That is, for every function $f : [\kappa]^2 \rightarrow 2$ (i.e., function on increasing pairs), there is a set $H \subset \kappa$ such that $|H| = \kappa$ and $f'' [H]^2 = c$.

2. It has the **tree property**: it is inaccessible and every tree of height $\kappa$ whose levels have cardinality $< \kappa$ has a branch of cardinality $\kappa$.

3. For every transitive set $M$ of size $\kappa$ with $\kappa \in M$, there is a transitive set $N$ and an embedding $j : M \rightarrow N$ with critical point $\kappa$.

4. Whenever $M$ is a set containing at most $\kappa$-many subsets of $\kappa$, then there is a $\kappa$-complete nonprincipal filter $F$ measuring every set in $M$.

5. It is uncountable and every $\kappa$-satisfiable theory in an $L_{\kappa, \kappa}$ language of size at most $\kappa$ is satisfiable.

6. $\kappa$ is $\Pi^1_1$-indescribable.
Weakly Compact (continued)

Proof.

For example: $5 \Rightarrow 2$: Let $\kappa$ be inaccessible and $\mathcal{L}_{\kappa,\omega}$ satisfy weak compactness. Suppose $(T, <)$ is a tree of height $\kappa$ and each level has size $< \kappa$. Take $\mathcal{L}_{\kappa,\omega}$ to be the language with a unary predicate $B$ and constant symbols $c_x$ for all $x \in T$. Specify $\Sigma$ as the following set of sentences:

- $\neg (B(c_x) \land B(c_y))$ for all $x, y \in T$ which are incomparable
- $\bigvee_{x \in T_{\alpha}} B(c_x)$ for all $\alpha < \kappa$ and $T_{\alpha}$ the $\alpha$-th level of $T$

Then $\Sigma$ says $B$ is a branch in $T$ of length $\kappa$. For $S \subset \Sigma$ with $|S| < \kappa$, $S$ is satisfiable. This implies $\Sigma$ is satisfiable which means $T$ has a branch of length $\kappa$. 

$\square$
Zero Sharp

Definition
A set $I \subseteq \kappa$ is a set of indiscernibles for a model $\mathcal{A}$ if $\forall n \in \omega$ and $\varphi(x_1, \ldots, x_n)$, $\mathcal{A} \models \varphi[\alpha] \iff \mathcal{A} \models \varphi[\beta]$ whenever $\alpha, \beta$ are increasing sequences from $I$.

Hence indiscernibles are indistinguishable by any first-order formula.

Definition
$0\# \exists$ exists:

- If $\kappa, \lambda$ are uncountable cardinals with $\kappa < \lambda$, then $(L_\kappa, \in) \preccurlyeq (L_\lambda, \in)$.
- There is a unique club class of ordinals $I$ containing all uncountable cardinals such that $\forall \kappa$ uncountable:
  - $|I \cap \kappa| = \kappa$
  - $I \cap \kappa$ is a set of indiscernibles for $(L_\kappa, \in)$
  - $\forall a \in L_\kappa$ is definable in $(L_\kappa, \in)$ from $I \cap \kappa$. 
Consequences of “0# Exists”

Proposition
0# exists ⇔ ∃ non-trivial elementary embedding of L into L.

Proposition
Every uncountable cardinal (of V) is inaccessible in L.

Proof.
L |= $\aleph_1$ regular ⇒ L |= $\aleph_\alpha$ regular ∀ $\alpha \geq 1$. Also, L |= $\aleph_\omega$ strong limit so that L |= $\aleph_\alpha$ strong limit ∀ $\alpha \geq 1$. Hence, every $\aleph_\alpha$ is inaccessible in L.

Proposition
0# exists ⇒ V ≠ L

Proof.
STS the set of all constructible reals is countable. For $\alpha \geq \omega$, $V_\alpha \cap L$ is definable in L from $\alpha$. This implies $V_\alpha \cap L$ is also definable from $\alpha$ in $L_\kappa$ where $\kappa$ is the least cardinal > $\alpha$. ⇒ $V_\alpha \cap L \subseteq L_\beta$ for some $\beta$ such that $|\alpha| = |\beta|$. But $|L_\beta| = \beta$. Since all reals are in $V_{\omega+1}$, all constructible reals are in $V_{\omega+1} \cap L \subseteq L_\beta$ for $\beta$ countable.
Definition
\( \kappa \) is **Ramsey** if \( \kappa \rightarrow (\kappa)^{<\omega}_2 \).

Theorem
(Silver) \( \exists \kappa \) Ramsey \( \Rightarrow 0^\# \) exists.
Measurables

Definition
An uncountable cardinal $\kappa$ is **measurable** if there exists a $\kappa$-complete nonprincipal ultrafilter $U$ on $\kappa$.

Proposition

$\kappa$ is measurable iff there exists a nontrivial elementary embedding $j : V \rightarrow M$ into some transitive model $M$ with $cp(j) = \kappa$.

Proof.

($\Leftarrow$) Assume there exists an elementary embedding $j : V \rightarrow M$ with $cp(j) = \kappa$. Set $\mu = \{A \subseteq \kappa : \kappa \in j(A)\}$. Note:

- $\kappa \in j(\kappa) \Rightarrow \kappa \in \mu$
- For $\beta < \kappa$ and $\langle A_\alpha \in \mu : \alpha \in \beta \rangle$, $j'' \langle A_\alpha : \alpha \in \beta \rangle = \langle j(A_\alpha) : \alpha \in \beta \rangle$ since $j(\alpha) = \alpha \forall \alpha < \kappa$. This implies $j(\bigcap \langle A_\alpha : \alpha \in \beta \rangle) = \bigcap \{j(A_\alpha) : \alpha < \beta\} \neq \emptyset$ since $\kappa \in j(A_\alpha)$. So $\mu$ is $\kappa$-complete.
- If $A \subseteq \kappa$ and $A \notin \mu$, then $\kappa \notin j(A)$. This implies $\kappa \in j(\kappa \setminus A)$ and, hence, $\kappa \setminus A \in \mu$ so that $\mu$ is maximal.
- Finally, $\exists \alpha (\{\alpha\} \in \mu)$ since $j(\alpha) = \alpha \forall \alpha < \kappa$.

Conclude $\mu$ is a $\kappa$-complete nonprincipal ultrafilter on a measurable cardinal $\kappa$.

($\Rightarrow$) Since the ultrapower embedding $j : V \rightarrow M$ by a measure $\mu$ on $\kappa$ is nontrivial.
Measurables (continued)

Proposition
If $\kappa$ is measurable, then it is the $\kappa^{th}$ Mahlo cardinal.

Theorem
If there is a measurable cardinal, then $V \neq L$.

Proof.
Assume $V = L$ and measurable cardinals exist. Let $\kappa$ be the least measurable. Let $\mu$ be a nonprincipal $\kappa$-complete ultrafilter on $\kappa$ and let $j : V \to M$ be the associated elementary embedding. Recall $j(\kappa) > \kappa$. Since $V = L$, the only transitive model containing all the ordinals is the universe itself. Hence, $V = M = L$. $j$ elementary implies $M \models j(\kappa)$ is the least measurable. But $j(\kappa) > \kappa$, a contradiction. \qed
Proposition

\( \kappa \) measurable implies \( \kappa \) is \( \Pi^2_1 \)-indescribable.

Proof.

Let \( \kappa \) be measurable and \( (V_\kappa, \in, U) \models \sigma \) for \( \sigma \in \Pi^2_1(\in, U) \). Then \( \sigma = \forall X \varphi(X) \) with \( X \) third order and \( \varphi \) containing only second and first order quantifiers. Let \( \overline{\varphi} \) be the first order sentence obtained from \( \varphi \) by replacing first order quantifiers by restricted quantifiers \( \forall x \in V_\kappa \) and \( \exists x \in V_\kappa \). Then \( \forall X \subset V_{\kappa+1} (V_{\kappa+1}, \in, X, V_\kappa, U) \models \overline{\varphi} \).

Let \( M = \text{Ult}_\mu(V) \) for \( \mu \) a normal measure on \( \kappa \). Since \( V^M_{\kappa+1} = V_{\kappa+1} \), this also holds in \( M \). In the ultrapower, \( V_\kappa \) is represented by the function \( \alpha \mapsto V_\alpha \) and \( \alpha \mapsto V_{\alpha+1} \) represents \( V_{\kappa+1} \) and \( \mu \) is \( \alpha \mapsto \mu \cap V_\alpha \). So that for almost all \( \alpha \), \( \forall X \subset V_{\alpha+1} (V_{\alpha+1}, \in, V_\alpha, U \cap V_\alpha) \models \overline{\varphi} \). Translating back into the third order language, \( (V_\alpha, \in, U \cap V_\alpha) \models \sigma \) for almost all and, hence, for some \( \alpha < \kappa \). \( \square \)
Supercompact

Definition

κ is supercompact if ∀θ ∈ ON there is an elementary embedding j : V → M with M a transitive class, such that \( cp(j) = \kappa \) and \( M^\theta \subset M \).

Alternative characterizations:

1. κ is supercompact if ∀A |A| > κ there exists a normal measure on \( P_\kappa (A) \). A measure is normal if it is closed under diagonal intersections \( \triangle X_\alpha = \left\{ \beta < \delta : \beta \in \bigcap_{\alpha<\beta} X_\alpha \right\} \neq \emptyset \) for any collection of fewer than κ elements of the measure.

Reflection properties:

1. If κ is supercompact, then \( V_\kappa \prec_2 V \), i.e., all \( \Sigma_2 \) formulas are absolute.

2. So, if GCH holds below a supercompact κ, then a bijection between \( \mathcal{P}(\alpha) \) and \( \alpha^{++} \) would be a \( \Sigma_2 \) witness of the failure of GCH and would, hence, occur below κ.
Supercompact Consequences

Proposition

*If Ϝ is supercompact then Ϝ is the Ϝ-th measurable cardinal.*

Proof.

Let λ = 2^Ϝ and j : V → M witness the λ-supercompactness of Ϝ. Define D = \{X : Ϝ ∈ j(X)\} and let j_D : V → Ult_D be the corresponding elementary embedding. Set h : Ult_D → M be the elementary embedding h([f]_D) = (jf)(Ϝ). Note that h(Ϝ) = Ϝ.

Now, P(Ϝ) ⊂ M and every subset of M of size λ is in M; hence every U ⊂ P(Ϝ) is in M and it follows that in M, Ϝ is a measurable cardinal. Since h is elementary and h(Ϝ) = Ϝ, we have Ult_D \models Ϝ is a measurable cardinal.
Extendibles

**Definition**

$k$ is **extendible** if $\forall \alpha > k$ there is $\beta \in \mathbb{ON}$ and an elementary embedding $j : V_\alpha \to V_\beta$ with $cp(j) = k$.

**Fact**

*Reflection property*: If $k$ is extendible, then $V_k \prec_3 V$ (all $\Sigma_3$ formulas are absolute).

**Claim**

If $k$ is extendible, then $k$ is supercompact.
Vopenka’s Principle

Definition
For any language $L$ and proper class $\mathcal{C}$ of $L$-structures, there are distinct structures $M, N \in \mathcal{C}$ and an elementary embedding $j : M \to N$. An inaccessible cardinal $\kappa$ is Vopenka iff $V_\kappa$ satisfies Vopenka’s principle where “proper class” of $V_\kappa$ is interpreted to be a subset of $V_\kappa$ of size $\kappa$.

Proposition
If Vopenka’s Principle holds, then there exists an extendible cardinal.

Proof.
Let $A$ be the class of all limit ordinals $\alpha$ with $\text{cf} (\alpha) = \omega$ and such that for every $\kappa < \alpha$, if $V_\alpha \models (\kappa \text{ extendible})$, then $\kappa$ is extendible and for $\kappa < \gamma < \alpha$, if there is an elementary embedding $j : V_\gamma \to V_\delta$ with critical point $\kappa$, then $V_\alpha \models \text{(there is an elementary embedding)}$. Using the RP, $A$ is a proper class. Let $C$ consist of the models $(V_{\alpha+1}, \in)$ for $\alpha \in A$. Then, by $VP$, $\exists \alpha, \beta \in A$ and an elementary embedding $j : V_{\alpha+1} \to V_{\beta+1}$. Since $j(\alpha) = \beta$, it moves some ordinal. Hence, its critical point is measurable and so it is not $\alpha$ (which has cof $\omega$). Let $\kappa$ be the critical point. Now $V_\alpha \models (\kappa \text{ extendible})$ because for every $\gamma < \alpha$, $j \upharpoonright V_\gamma$ reflects to a witness to extendibility. By definition of $A$, $\kappa$ is extendible.
Ridiculously Strong Hypothesis

Definition

Rank-into-rank axioms:

1. \( I_3: \exists j : V_\lambda \rightarrow V_\lambda \)
2. \( I_2: \exists j : V \rightarrow M \) such that \( V_\delta \subseteq M \) for some \( \delta > cp(j) \) with \( j(\delta) = \delta \).
3. \( I_1: \exists \delta \) and \( \exists j : V_{\delta+1} \rightarrow V_{\delta+1} \)
4. \( I_0: \exists \delta \) and \( \exists j : L(V_{\delta+1}) \rightarrow L(V_{\delta+1}) \) with \( cp(j) < \delta \).

Remark

#4 was developed by Woodin to prove the axiom of determinacy in \( L(\mathbb{R}) \). It turned out to be much more than was needed.
Rated XXX (a.k.a. Inconsistency)

Definition

\[(0=1) \exists j : V \to V.\]

Theorem

\[(\text{Kunen '71}) (\text{AC}) \text{ If } j : V \to M, \text{ then } M \neq V.\]

Remark

The proof also relies on the axiom of foundation. In some anti-foundation frameworks, there are provable, nontrivial elementary embeddings of \( V \) into \( V \). It is not known whether this result can be proven in ZF.