SOTS: Complex Geometry and Kähler Manifolds
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Abstract: Complex geometry is the complex analogue of differential geometry, replacing \( \mathbb{R} \) with \( \mathbb{C} \) and “smooth functions” with “holomorphic functions.” Kähler manifolds are a type of complex manifold which locally approximate Euclidean space to order 2. A subset of them can be explicitly described by polynomials, providing a beautiful link to algebraic geometry. In this talk we will introduce complex manifolds, then discuss Kähler manifolds and some of their properties. Finally we will state the Kodaira Embedding Theorem, a complex analogue of the Whitney Embedding Theorem for real manifolds.

1 Motivation

Complex geometry is the geometry of manifolds which locally look like open sets in \( \mathbb{C}^n \), (instead of \( \mathbb{R}^n \)), where transition functions are holomorphic. Many differential geometric-notions have counterparts in complex geometry. Complex manifolds are more rigid than real manifolds, since for example holomorphic functions are more restrictive than smooth functions. On the other hand, some complex manifolds can be described explicitly in terms of polynomials, which leads us into the realm of algebraic geometry.

The motivating question for this talk is: do we have an analogue of the Whitney Embedding theorem in complex geometry? This says that every smooth real manifold can be embedded into \( \mathbb{R}^N \) for some \( N \). If we try to embed a compact complex manifold into \( \mathbb{C}^N \), then the coordinate functions will give holomorphic functions on the compact manifold, so must be
constant. So instead we look at when a complex manifold can be embedded into projective space $\mathbb{CP}^N$. It turns out that a special class of Kähler manifolds called “Hodge manifolds” are embeddable into projective space.

2 Background

2.1 Coordinates

Definition 1. A complex manifold $M$ of dimension $n$ is a (real) differentiable manifold of dimension $2n$ which has an equivalence class of systems of holomorphic charts.

A system of holomorphic charts means: a covering of $M$ by open sets $U_i \subset M$ and maps $\phi_i : U_i \to \mathbb{C}^n$ where $\phi_i$ is a homeomorphism onto an open set in $\mathbb{C}^n$, such that

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is holomorphic. The charts are the pairs $(U_i, \phi_i)$. The collection of charts gives an atlas, and we say that two atlases are equivalent if the union of their charts forms a system of holomorphic charts. An equivalence class of atlases gives a complex structure on $M$, which is then a complex manifold.

Charts define coordinates. Suppose $\phi : U \to \mathbb{C}^n$ is some chart centered around a point $p \in U$ (meaning $p$ maps to the origin.) Then $\phi$ can be written as local coordinates $(z_1, \ldots, z_n)$, where each $z_i$ is a holomorphic function on $U$. These are complex coordinates on $M$. Note that if $M$ is a complex $n$-dimensional manifold, it can be realized as a real $2n$-dimensional manifold with coordinates $x_i, y_i$ coming from $z_j = x_j + iy_j$.

Example: Complex projective space. $\mathbb{CP}^n$ is the set of equivalence classes denoted $[z_0 : \ldots : z_n]$, where the $z_i$ are complex numbers. We have an open cover given by $U_i = \{[z_0 : \ldots : z_n] | z_i \neq 0\}$. We have bijective maps $\phi_i : U_i \to \mathbb{C}^n$ given by $[z_0 : \ldots : z_n] \mapsto (z_0/z_i, \ldots, z_i/z_i, \ldots, z_n/z_i)$. It remains to show the transition functions are holomorphic. These are given by $\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$. Let $w_k$ be the coordinates on $U_j$ and assume $i < j$. Then

$$\phi_{ij} : (w_0, \ldots, \hat{w}_j, \ldots, w_n) \xrightarrow{\phi_j^{-1}} [w_0 : \ldots : 1 : \ldots : w_n] \xrightarrow{\phi_i} \left(\frac{w_0}{w_i}, \ldots, \frac{\hat{w}_j}{w_i}, \ldots, \frac{1}{w_i}, \ldots, \frac{w_n}{w_i}\right)$$

Since $w_i \neq 0$ on $\phi_j(U_i \cap U_j)$, we see that the coordinate functions of $\phi_{ij}$ are of the form $w_k/w_i$ or $1/w_i$, which are holomorphic on the domain of $\phi_{ij}$. So $\mathbb{CP}^n$ is a complex manifold.
2.2 Tangent space

The real tangent space $T_R M$ has local frame $\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \}_{1 \leq i \leq n}$, defined at a point $p$ as the space of $\mathbb{R}$-linear derivations of germs of smooth functions at $p$.

Definition 2. We define the holomorphic tangent bundle $T^{1,0} M$ at a point $p$ to be the vector space of $\mathbb{C}$-linear derivations of germs of holomorphic functions at $p$, i.e. of $\mathcal{O}_{M,p}$. It has local frame $\{ \frac{\partial}{\partial z_i} \}_{1 \leq i \leq n}$. We define the antiholomorphic tangent bundle $T^{0,1} M$ at $p$ to be given by $\mathbb{C}$-linear derivations of germs of antiholomorphic functions at $p$ (we say $f$ is antiholomorphic if $\overline{f}$ is holomorphic.) It has local frame $\{ \frac{\partial}{\partial \bar{z}_i} \}_{1 \leq i \leq n}$.

What's the relation between $T_R M$ and $T^{1,0} M$? It turns out they are isomorphic as $2n$-dimensional real bundles. In particular, multiplication by $i$ on the latter corresponds with an “almost complex structure” on the former. It will be useful to show that $T^{1,0} M \cong T_R M$ since then to do geometry on $M$, we can do it purely on the holomorphic tangent bundle.

$T^{1,0} M \cong T_R M$. Write $T_C M$ for $T_R M \otimes \mathbb{C}$. We will see that the holomorphic tangent bundle $T^{1,0} M$ lives inside $T_C M$ as an eigenspace. Define the endomorphism

$$J : T_R M \to T_R M \quad \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i} \quad \frac{\partial}{\partial y_i} \mapsto - \frac{\partial}{\partial x_i}$$

Then $J^2 = -1$ and is called an almost complex structure. It extends $\mathbb{C}$-linearly to $T_C M$ and splits $T_C M$ into $+i$ and $-i$ eigenspaces, which we denote $T^{1,0} M \oplus T^{0,1} M$.

The isomorphism $T^{1,0} M \cong T_R M$ comes from the map $T_R M \hookrightarrow T_C M \to T^{1,0} M$. We need to know how to project onto the eigenspace $T^{1,0} M$, i.e. how to write an element $X \in T_C M$ in terms of elements of the two eigenspaces.

$$X = \frac{1}{2} (X - iJ(X)) + \frac{1}{2} (X + iJ(X))$$

A quick check applying $J$ to the two summands shows they lie in the desired eigenspaces. So

$$T_R M \hookrightarrow T_C M \to T^{1,0} M \quad \frac{\partial}{\partial x_i} \mapsto \frac{1}{2} \left( \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right) \quad \frac{\partial}{\partial y_i} \mapsto \frac{1}{2} \left( \frac{\partial}{\partial y_i} + i \frac{\partial}{\partial x_i} \right)$$
How does this relate to $\frac{\partial}{\partial z_i}$? Recall that $z_j = x_j + iy_j$ so on 1-forms $dz_j = dx_j + idy_j$. This basis should be dual to $\frac{\partial}{\partial z_i}$; in other words we want $dz_j \left( \frac{\partial}{\partial z_i} \right) = \delta_{ij}$. A calculation shows that

$$\frac{\partial}{\partial z_i} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right)$$

We know on $T\mathbb{R}M$ that $J : \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$. Hence $J$ sends $\frac{1}{2} \left( \frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i} \right) \mapsto \frac{1}{2} \left( \frac{\partial}{\partial y_i} + i \frac{\partial}{\partial x_i} \right)$. Therefore on $T^{1,0}M$, $J$ is

$$\frac{\partial}{\partial z_i} \mapsto i \frac{\partial}{\partial z_i}$$

In summary: $T^{1,0}M \cong T\mathbb{R}M$ via the isomorphism sending $\frac{\partial}{\partial z_i} \mapsto \frac{\partial}{\partial z_i}$ and where $J$ on $T\mathbb{R}M$ corresponds to multiplication by $i$ on $T^{1,0}M$. (To show we have an isomorphism as vector bundles, we also need to show the transition functions are the same.) Now we can define metrics and curvature by looking at the holomorphic tangent plane.

### 2.3 Metric

In differential geometry we have a Riemannian metric, which is a symmetric positive-definite bilinear form on each tangent space. The corresponding notion in complex geometry is a Hermitian metric. It is a Hermitian positive-definite bilinear form on $T^{1,0}M$. Locally in coordinates it has the form:

$$h = \sum_{i,j=1}^{n} h_{ij} dz_i \otimes d\bar{z}_j$$

where $h_{ij} = \overline{h_{ji}}$. So what’s the relation between these two types of metrics?

**Definition 3.** Suppose we’re given a Riemannian metric $g$ on a complex manifold $M$ (considered as a real smooth manifold) which is compatible with the almost complex structure $I$, meaning $g(I(u), I(v)) = g(u, v)$ for all vectors $u, v$. We define the Kähler form $\omega$ to be the 2-form

$$\omega(u, v) := g(I(u), v)$$

1) We get a Hermitian metric on $T\mathbb{R}M$ via $h := g - i\omega$.

2) We get a Hermitian metric on $T^{1,0}M$ via extending $g$ to a Hermitian form $g_C$ on $T\mathbb{C}M$, i.e. $g_C(u \otimes \lambda, v \otimes \mu) = \lambda \overline{\mu} \cdot g(u, v)$. Then we show the direct sum of eigenspaces is an orthogonal decomposition in $T^{1,0}M \oplus T^{0,1}M$. 


Under the isomorphism $T_R M \cong T^{1,0} M$, $\frac{1}{2} h$ corresponds with $g_C |_{T^{1,0} M}$. This means that

$$\frac{1}{2} h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = g_C \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right)$$

If we write $h_{ij} = h \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)$, then the K"{a}hler form can be written in local coordinates as:

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\bar{z}_j$$

2.4 Refresher on differential forms

- Given a vector space $W$, we define

$$\wedge^n W = W \otimes \ldots \otimes W / I$$

where we tensor $W$ $n$ times and quotient by the ideal $I$ generated by elements of the form $v \otimes w + w \otimes v$.

- Let $\mathcal{A}^{p,q}(M)$ denote the global sections of the bundle $\wedge^p T^{1,0} M^* \otimes \wedge^q T^{0,1} M^*$. These are called $(p, q)$ forms. So the K"{a}hler form is a $(1, 1)$ form.

- The exterior derivative $d = \partial + \overline{\partial}$ is defined on smooth functions by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i + \frac{\partial f}{\partial \overline{z}_i} d\overline{z}_i = \partial f + \overline{\partial} f$$

We extend this to general $(p, q)$ forms using the Leibniz rule, as in the real case in differential geometry. For example on 1-forms we have $d(f dg) = df \wedge dg + f d^2 g = df \wedge dg$ since $d^2 = 0$.

- The fact that $d^2 = 0$ is a calculation and arises from the fact that we are dealing with alternating forms, so interchanging two elements introduces a negative sign.

- $\partial$ maps $(p, q)$ forms to $(p + 1, q)$ forms and $\overline{\partial}$ maps $(p, q)$ forms to $(p, q + 1)$ forms.
2.5 Curvature

A connection is a way to differentiate vector fields in the direction of other vector fields. The distinguished connection in differential geometry is the Levi-Civita connection $\nabla$. It is the unique one which is torsion-free (meaning $\nabla_X Y - \nabla_Y X = [X, Y]$ for all vector fields $X, Y$) and compatible with the Riemannian metric $\langle , \rangle$ (meaning $d \langle X, Y \rangle = \langle \nabla_X Y, \rangle + \langle X, \nabla Y \rangle$).

The distinguished connection in complex geometry is the Chern connection $D$. It is the unique one which is compatible with the Hermitian metric and compatible with the complex structure on a holomorphic vector bundle. The latter condition means that $D$ “determines the holomorphic structure.” If we write $D = D^{1,0} + D^{0,1}$ as the sum of operators mapping to the sheaf of $T^{1,0}M$-valued $(1,0)$ and $(0,1)$ forms, then “compatible with the complex structure” means $D^{0,1} = \overline{\partial}_{T^{1,0}M}$. One way to see this as natural is that a smooth function $f$ is holomorphic if and only if $\overline{\partial} f = 0$, since this condition is equivalent to satisfying the Cauchy-Riemann equations in each coordinate. So $D^{0,1} = \overline{\partial}_{T^{1,0}M}$ means that $D^{0,1}$ kills holomorphic sections.

Remark 4. We can think of curvature as measuring how curved a manifold is. We can use the connection to differentiate twice and if we don’t get zero, the manifold is curved instead of flat.

Remark 5. When a manifold is Kähler, which we will define next, the Chern connection and Levi-Civita connection are the same via the isomorphism $T^{1,0}M \cong T_{\mathbb{R}}M$. The proof amounts to showing that the Chern connection is torsion-free, which will be ensured by the condition of being Kähler. So real and complex geometry interact well together in the Kähler case. In particular, Riemannian curvature and Chern curvature coincide. So if we have a Ricci flat manifold, this tells us the first Chern class vanishes, because the first Chern class $c_1(M) = \frac{i}{2\pi} [\rho]$ where $\rho$ is the trace of Chern curvature $\Theta$.

3 Kähler manifolds

The formal definition:

Definition 6. A Kähler manifold is a complex manifold whose Kähler form $\omega$ is $d$-closed, meaning $d\omega = 0$.

What does this mean geometrically? An equivalent condition for being a Kähler manifold is that the metric $g$ locally looks like the flat Euclidean metric to the first degree. The condition is that $g$ “osculates to order 2” to the Euclidean metric. The standard metric is $\sum_{i=1}^{n} dz_i \otimes dz_i$. “Osculate to order 2” means that about any given point $p$, there exist coordinates $z_i$ such that we can express the metric as $\sum_{i,j} (\delta_{ij} + O(2)) dz_i \otimes dz_j$.
Lemma 7. Let $M$ be a complex manifold with Hermitian metric $g$. Then $g$ is Kähler iff it osculates to order 2 to the Euclidean metric at every point.

Idea of proof. ($\Leftarrow$): Given a metric in the above form, we can directly calculate that $d$ of it is zero.

($\Rightarrow$): First we write the metric in coordinates so that it equals $\delta_{ij}$ at a specific point $p$ (choose a basis so that the inner product is the identity matrix on the vector space given by the tangent bundle at $p$). Then add its first and second order terms in these local coordinates.

$$
\omega = \frac{i}{2} \sum_{i,j,k} (\delta_{ij} + a_{ijk} z_k + a_{ij k} \overline{z}_k + O(|z|^2)) dz_i \wedge d\overline{z}_j
$$

We then show that $d\omega = 0$ if and only if the first order terms $a_{ijk}$ and $a_{ij k}$ are zero. \qed

Example: projective space continued. There is a metric on $\mathbb{CP}^n$ called the Fubini-Study metric, which we can define from the corresponding Kähler form. We define the form in local coordinates $z_1, \ldots, z_n$ as

$$
\omega_{FS} = \frac{i}{2} \partial \overline{\partial} \log(1 + \sum_{i=1}^{n} |z_i|^2)
$$

Note that we can see that $d\omega_{FS} = 0$ immediately. This follows from the fact that $d = \partial + \overline{\partial}$, $\partial^2 = \overline{\partial}^2 = 0$, and $\partial \overline{\partial} = -\overline{\partial} \partial$. There are several things that still need to be checked. We’ve only defined this locally so we need to show that the definition is compatible with a choice of different charts. We also need to show that $\omega_{FS}$ corresponds to a Hermitian metric, i.e. something which is positive definite and Hermitian at each point. Note that the unitary group $U(n+1)$ acts transitively on $\mathbb{CP}^n$ and it leaves $\omega_{FS}$ invariant, so we only need to check these at a point.

Remark 8. It is a fact that $d$ commutes with restriction maps, so since $\mathbb{CP}^n$ is Kähler, any submanifold of complex projective space is Kähler.

Another example. Complex tori are also Kähler manifolds. Recall that on $\mathbb{C}^n$ we have the metric $ds^2 = \sum_i dz_i \otimes d\overline{z}_i$. This is translation invariant, i.e. invariant under adding constants to the $z_i$. Its Kähler form is clearly closed, since $d^2 = 0$. So it descends to a Kähler metric on $\mathbb{C}^n / \Lambda$ for any rank $2n$ discrete lattice $\Lambda \subset \mathbb{C}^n$, hence complex tori are Kähler.

A non-Kähler example. Consider the Hopf manifold obtained by $M = \mathbb{C}^n \setminus \{0\} / \sim$ where $(z_1, \ldots, z_n) \sim (w_1, \ldots, w_n)$ if and only if there exists an integer $k$ such that $z_i = 2^k w_i$ for all $i$. We assume $n \geq 2$. Topologically, $M$ is homeomorphic to $S^1 \times S^{2n-1}$, thus there is no cohomology in degree 2. That is, $H^2(M, \mathbb{C}) = 0$. In particular, there cannot exist a Kähler form corresponding to a Kähler metric, since it would be a 2-form. So $M$ is not Kähler.
4 Kodaira Embedding Theorem, Chow’s Theorem

When can a compact complex manifold be embedded into \( \mathbb{CP}^n \)? The answer is: for certain Kähler manifolds called Hodge manifolds.

**Definition 9.** A Hodge manifold is a Kähler manifold such that the Kähler form \( \omega \) is integral, i.e. it defines an element of \( H^{1,1}(M) \cap H^2(M, \mathbb{Z}) \).

**Example.** Any Riemann surface (a complex manifold of complex dimension 1) is a Hodge manifold. \( H^2(M, \mathbb{C}) \) is one dimensional, generated by some \( d \)-closed form \( \omega \). We can then scale \( \omega \) so it defines an integral class.

One can show that the Fubini-Study metric gives rise to an integral cohomology class, so anything embeddable in projective space is a Hodge manifold. The converse is Kodaira’s Embedding Theorem.

**Theorem 10** (Kodaira Embedding Theorem). *A compact complex manifold \( M \) is embeddable in complex projective space if and only if it admits a Kähler form which is closed and integral.*

Another version of the theorem is:

**Theorem 11.** Let \( M \) be a compact complex manifold and \( L \to M \) a positive line bundle. Then \( \exists k_0 \) such that \( \forall k \geq k_0 \) there is a well-defined embedding of \( M \) into projective space, \( \iota_{L^k} : M \to \mathbb{CP}^N \) for some \( N \).

**Definition 12.** A positive line bundle is a holomorphic vector bundle of rank 1 such that we can write its Chern curvature form as \( \frac{i}{2\pi} \Theta \) with positive-definite coefficients.

The embedding \( \iota_L \) is via sections of the positive line bundle \( L \). Let \( H^0(M, L) \) be the vector space of sections of \( L \). There is a result which states this is finite dimensional. Let \( s_0, \ldots, s_N \) be a basis. The embedding \( \iota_L \) is defined by:

\[
\iota_L : M \ni p \mapsto [s_0(p) : \ldots : s_N(p)] \in \mathbb{CP}^N
\]

Note that a line bundle is locally trivial so under this trivialization, it makes sense to talk about the complex number given by a section \( s \) of \( L \) at a point \( p \), i.e. \( s(p) \). To show the map is well-defined, we note that choosing a different trivialization results in multiplying all the \( s_i \) by the same transition function, a non-zero number for a line bundle. Hence we define the same point in projective space. Some other things that need to be shown:

- At no \( p \) do all \( s_i \) vanish.
- The map \( \iota_{L^k} \) is injective.
• The derivative $dL^k$ is non-zero everywhere.

The proof of these uses the cohomology long exact sequence arising from the exponential short exact sequence, as well as blow-ups of manifolds. The relation of the first formulation to the second formulation of the theorem is that if $M$ is Kähler and the Kähler form $[\omega] \in H^2(M, \mathbb{Z})$, then there exists a positive line bundle $L$ on $M$ corresponding to $\omega$ under the cohomology long exact sequence. (We use that the class of line bundles $\text{Pic}(M)$ is isomorphic to $H^1(M, \mathcal{O}_M^*)$.)

We can say more once we have an embedding into projective space, using the following theorem.

**Theorem 13** (Chow’s Theorem). *An analytic subvariety of projective space is algebraic, meaning it can be described by polynomials.*

So in fact any compact Hodge manifold can be described explicitly by polynomials in $\mathbb{C}P^N$.

**Corollary 14.** *Every compact Riemann surface is algebraic.*

**References**
