Abstract

In complex geometry Kodaira's theorem tells us that on a Kähler manifold sufficiently high powers of positive line bundles admit global holomorphic sections. Donaldson's divisor theorem is a symplectic analogue of this theorem, proving the existence of asymptotically holomorphic sections for positive line bundles on a symplectic manifold with almost complex structure. In this talk, we will outline the methods used in Donaldson's proof, and state some applications of Donaldson's divisor theorem to symplectic geometry.

0.1 Complex Analogues

We start with a few recollections from complex geometry.

Definition 0.1 Ample Line Bundle

Let M be a complex manifold. A complex line bundle $L \to M$ is called *ample* if there exists k such that $L^{\otimes k}$ has enough sections to set up an embedding to $\mathbb{CP}^n$.

Ample line bundles come with a very strict geometric condition on them:

Definition 0.2 Positive Line Bundle

A complex line bundle $L \to M$ is called *positive* if it's Chern class $c_1(L)$ is represented by some Kähler metric.

Proposition 0.3

All ample line bundles are positive.

Proof. The Fubini-Study metric is the first Chern class of $\mathcal{O}(1)$, and $L^{\otimes k}$ is the pullback of $\mathcal{O}(1)$. ∎

In particular, M is Kähler. We also have the converse statement:

Proposition 0.4

All positive line bundles are ample.

Proof. This is the content of the Kodaira embedding theorem. ∎

Theorem 0.5 Kodaira Embedding Theorem

If (M, ω) is a compact Kähler manifold and L is a line bundle with curvature form ω, then M is projective, and ω is a positive integer multiple of the pullback of the Fubini-Study Metric.

In addition, we get the following corollary

Corollary 0.6

Let h be the first Chern class of a positive line bundle on M. Then for sufficiently large k, there exists complex submanifold N such that N realizes the Poincaré dual of kh.

This says that every positive line bundle has a power represented by a divisor. We would like to reformulate these statements for symplectic manifolds with compatible pseudoholomorphic structure.

0.2 Symplectic Goals

The goal of Donaldson's paper [3] is to give us a related theory in the case of symplectic manifolds. While we cannot talk about holomorphic functions in this setting, we can talk about pseudoholomorphic functions coming from an compatible almost complex structure J on M.

Definition 0.7

Let $E \to M$ be a complex vector bundle compatible with an almost complex structure on M. Let ∇ be the natural connection arising from the symplectic structure. Then ∇ decomposes as $\partial + \bar{\partial}$, complex linear and anti-linear parts.

- We say that a map is pseudoholomorphic $s : M \to \mathbb{C}$ is pseudoholomorphic if $\bar{\partial}s = 0$.
- We say a sequence of sections s_k is **asymptotically holomorphic** if there exist constants $(C_p)_{p \in N}$ such that for all k and at every point in M, we have the following bounds.
  - $|s_k| \leq C_0$
  - $|\nabla^p s_k|_{g_k} \leq C_p$
  - $|\nabla^{p-1} \bar{\partial} s_k|_{g_k} \leq C_p k^{-1/2}$ for all $p \geq 1$.
- We say that a sequence of sections are **uniformly transverse** to 0 if there exists a constant $\eta > 0$ independent of k such that the sections s_k are η-transverse to 0 in the sense that everywhere

where $|s_k(x)| < \eta$, the linear map $\nabla s_k(x) : T_M \to (E_k)_x$ is surjective with right inverse of norm less than η^{-1} with respect to the metric g_k.

The main result is the following :

Theorem 0.8 Donaldson's Divisor Theorem

Let (M, ω) be a symplectic manifold with compatible almost-complex structure. Let $L \to M$ be a complex line bundle with Chern class $[\omega/2\pi]$. Then there exists a constant C such that for large enough k there is a section s of $L^{\otimes k}$ which is asymptotically holomorphic and uniformly transverse in the sense that

$$|\bar{\partial} s| < \frac{C}{\sqrt{k}} |\partial s|$$

on the zero locus of s.

While a symplectically holomorphic section does not give us a pseudo-holomorphic submanifold, it is enough to give a symplectic submanifold. The bound, in fact, gives us a sequence of symplectic submanifolds which are asymptotically pseudo-holomorphic, which is very surprising.

Lemma 0.9 [3]

Let (M, ω) be a compact symplectic manifold with integer symplectic form. Suppose $L \to M$ is a complex line bundle, and let s be a smooth section of L. Let W be the zero locus of s (a smooth submanifold of codimension 2). Let $\partial, \bar{\partial}$ be the complex linear and antilinear components of the connection ∇ coming from the symplectic and almost holomorphic structure. If $|\bar{\partial} s| < |\partial s|$ everywhere on W, then W is a symplectic submanifold.

The existence of such sections gives us something similar to the Kodaira embedding theorem.

Theorem 0.10 [1]

Let (M, ω) be a compact symplectic 4- manifold with integer symplectic form. There exists a map $f : X \to \mathbb{CP}^2$ which is "ϵ-branched covering."

Here, an ϵ-branched covering is an $\epsilon-$ approximate holomorphic map which is locally a diffeomorphism, a branched covering, or a cusp covering. The proof of 0.2 requires creating 3 approximately holomorphic sections of a line bundle.

1 Outline of Proof of 0.2

For a proof of this theorem , we follow [3] and use a simplification from [2] in our argument.
Donaldson's proof has 2 parts.

1. The first part is to do a local construction. This is a series of estimates that allows us to transfer to local Darboux coordinates, and create sections s of L^k so that $\bar{\partial} s$ is small everywhere on M. We won't be able to do this globally, but we'll create a bunch of local sections that have desired asymptotic holomorphic bounds. However, they will not be uniformly transverse (in fact, they will fail this requirement very badly).
2. The second section is a "global", and it shows that ∂s of these sections is not small on the zero set. We use an argument from **??** to get this estimate.

2 Local Bounds

First we construct local sections

Lemma 2.1

Given any point $x \in X$, for all large enough k, there exist asymptotically holomorphic section $s_{k,x}$ of $L^{\otimes k}$ such that

- $\sigma_{k,x} \geq c_0$ at every point of g_k radius 1 centered at x.
- The sections $\sigma_{k,x}$ have uniform Gaussian decay from x in C^2 norm.

We do this by taking a very specific coordinate chart:

Lemma 2.2

Let (M, ω) be a symplectic manifold, and k an integer. There exists a constant $c > 0$ and local Darboux coordinates $z_k^i : (X, x) \to (\mathbb{C}^n, 0)$ such that the following estimates hold uniformly in x and k at

> every point in the ball $B_{g_k}(x, c\sqrt{k})$:
> $$|z_k(y)| = O(\mathrm{dist}(x,y))$$
> $$|\bar\partial z_k(y)|_{g_k} = O\left(\frac{\mathrm{dist}_{g_k}(x,y)}{\sqrt{k}}\right)$$
> $$|\nabla^r \bar\partial z_k|_{g_k} = O\left(\frac{1}{\sqrt{k}}\right)$$
> $$|\nabla^r z_k|_{g_k} = O(1)$$

Idea of Proof. The idea is as we increase k, we cause the almost complex structure to look more and more like a complex structure. By Darboux's theorem we choose a chart around $x \in X$ called $\phi : B \to X$ such that $\phi^*(\omega)$ is the standard form on $B \subset \mathbb{C}^n$. We may suppose that all derivatives of ϕ are bounded and independent of x. We may also assume that the pullback of J agrees with the complex structure of B at the origin. We may relate the complex structure J_{std} to $\phi^* J$ by a map $\mu : \Lambda^{1,0}(B) \to \Lambda^{0,1}(B)$ so that the space of $\phi^* J$ complex forms is represented by the graph of μ. Since the structures agree at 0, we have that $\mu(0) = 0$.
We cannot in general make μ constantly 0– the obstruction to removing the first derivative of μ is the Nijenhius tensor. However, when we are willing to work on smaller and smaller balls, the bundle map μ looks closer and closer to being identically 0. We can express this as follows: given a scaling factor $\sqrt{k}$, we can get a new set of coordinates $\phi_{1/\sqrt{k}} = \frac{1/\sqrt{k}}{\phi}$. Then the bundle map $\mu_{1/\sqrt{k}}(z)$ corresponding to these new zoomed in coordinates satisfies bounds:

$$|\mu_{1/\sqrt{k}}(z)| \leq \frac{C|z|}{\sqrt{k}}$$
$$|\nabla \mu_{1/\sqrt{k}}(z)| \leq \frac{C}{\sqrt{k}}$$

■

So, we can see where at least where some of the dependence of these bounds on $\sqrt{k}$ are. But why are we using $\sqrt{k}$ here?
For this, we need to look at the relation between curvature and complex geometry. Equip B with the standard Kähler form
$$\omega_0 = \frac{i}{2}\sum_\alpha dz_\alpha d\bar z_\alpha$$
Then $\omega_0 = idA$ where
$$A = \frac{1}{4}\sum_\alpha z_\alpha d\bar z_\alpha - \bar z_\alpha dz_\alpha)$$
so that $-i\omega_0$ is the curvature form of the trivial complex line bundle over $\mathbb{C}^n$ with connection matrix A. This defines a $\bar\partial$ operator
$$\bar\partial A(f) = \bar\partial f + A^{0,1} f.$$
Now this complex vector bundle has a holomorphic section which decays rapidly to infinity in $\mathbb{C}^n$,
$$\sigma(z) = e^{-|z|^2/4}$$
With respect to this connection:
$$\bar\partial_A \sigma(z) = \frac{1}{4}\left(\sum z_\alpha d\bar z_\alpha - z\alpha d\bar z_\alpha) e^{-|z|^2/4}\right) = 0$$
and
$$\partial_A(\sigma(z) = \frac{1}{2}\left(\sum \bar z_\alpha dz_\alpha\right) e^{-|z|^2/4}$$
So, the positive curvature tensor ω_0 in these coordinates gives us a holomorphic section with exponential decay. Let's call this trivial bundle $\xi \to B$. If instead, we had the curvature tensor $k\omega_0$, we would have a section of $\xi^k \to B$ which would have norm $e^{-k|z|^2/4} = e^{|-\sqrt{k}z|^2/4}$. Aha! It appears that taking the kth tensor power of a bundle with positive curvature at least locally provides the same effect as applying a dilation of $\frac{1}{\sqrt{k}}$ to our coordinates. This gives us the basis of our second local estimate:

Sketch of Proof. We have a model for constructing these sections in standard coordinates. We just need to check how much are the derivates of these sections changed when we go from the standard complex structure to an almost complex structure instead. Let $\bar{\partial}_{J,k}$ denote the covariant derivative on $L^{\otimes}k \to (M,\omega)$ coming from ω and J. One can check that

$$\bar{\partial}_{J,k}(f) = (\bar{\partial}f + A^{0,1}f) + \mu_{\frac{1}{\sqrt{k}}}(\partial f, A^{1,0}f)$$

so that we have the following inequalities:

$$|\bar{\partial}_{J,k}f(z)| \le \frac{C|z|^2 e^{-|z|^2/4}}{\sqrt{k}}$$

$$|\nabla(\bar{\partial}_{J,k}f(z)| \le \frac{C(|z|+|z|^3)e^{-|z|^2}/4}{\sqrt{k}}$$

We would like to use the section σ that we constructed above and the new estimates to give the desired sections. However, the section σ is defined on all of $\mathbb{C}^n$. Let β_k be a bump function supported on a ball of radius $k^{1/6}$ so that

$$\nabla\beta_k = O(k^{\frac{-1}{6}})$$

Then one can check that $\beta_k\sigma$ satisfies all of the same inequalities as σ does, but is now supported on a ball of radius $k^{1/2}$.
Define the section $\sigma_{k,x}$ to be the pullback of the section $\beta_k\sigma$. ■

So now we have many little asymptotically holomorphic sections that exist around any given point. However, these sections are not uniformly transverse, because they are 0 almost everywhere. So , they are not yet analogous to the non-zero holomorphic sections that we are trying to imitate. Our goal now is to take all of these little sections, and tie them together somehow.

3 Building a Global Section

Our first attempt to build a global section is to simply add together the small local sections that we have. around points. We need to choose which points we are going to center our sections around for this construction. This choice needs to be reasonable and relies on the following lemma:

Lemma 3.1 There is a constant C such that for every k we can cover V by M sets which are g_k-unit balls with centers p_i such that for every point $q \in V$, we have

$$\sum_i d_k(p_i,q)^r e_k e_k(p_i,q) \le C$$

where $e_k(p,q) = e^{-d_k(p,q)^2/5}$ if $d_{p,q} \le k^{1/4}$ and 0 otherwise and $r = 0,1,2,3$.

Given a selection of points p_i, and a set of coefficients $\bar{w} = (w_1,\dots,w_M)$, define the section

$$s = s_{k,\bar{w}} - \sum_{i=1}^{M} w_i\sigma_{k,p_i}.$$

By the local bounds on the sections s_{k,p_i} and the choice of point placements, we have the following global bound on the section $s_{k.,\bar{w}}$.

Lemma 3.2 Let $|w_i| \le 1$ be a choice of coefficients. Then the section $s_{\bar{w},k}$ satisfies the following bounds :

$$|s| \le C$$

$$|\bar{\partial}Ls| \le \frac{C}{\sqrt{k}}$$

$$|\nabla\bar{\partial}_L s| \le \frac{C}{\sqrt{k}}$$

This gives us a section which is asymptotically holomorphic. However, we know nothing about the transversality of the section at the zero set of that section.

3.1 Transversality

The result that we would like to prove is

Proposition 3.3 There exists an $\epsilon > 0$ such that for all $k >> 0$ we can find ϖ such that $s_{\varpi,k}$ satisfies the transversality condition

$$|\partial s_{\varpi,k}\| > \epsilon$$

whenever $s_{w,s} = 0$.

The proof of this relies on some estimates on the sizes of perturbations of sections.

Theorem 3.4 (Main Technical Bound) Let B^+ be the ball of radius $\frac{3}{2}$ in $\mathbb{C}^n$, and let $f : B^+ \to C^m$ be a function. Assume that f satisfies the bounds

$$|f| \leq 1 \quad |\bar{\partial} f| \leq \eta$$

Then there exists a $w \in \mathbb{C}^m$ with $|w| \leq \delta < 1/4$ such that $f - w$ is η-transverse to 0 over the interior ball B of radius 1. The quantity η is dependent on δ as

$$\eta = \delta Q_p(\delta)$$

and $Q_p(\delta) = \log^{-1}(\delta)^{-p}$, where p depends only on m and n.

With this proposition we can start the proof of the transversality result as follows. Given the asymptotically holomorphic sections $s_{\varpi,k}$ as constructed above, consider the complex valued functions

$$f_{k,p_i} := s_k / s_{k,p_i}$$

defined in a small neighborhood of the p_i. By the lemma, there exists small constants w_k such that $f_k - w_k$ are uniformly transverse to 0. Multiplying again by s_{k,p_i}, we have that

$$s_k - w_k s_{k,p_i}$$

is uniformly transverse to 0 near x. This means that the transversality of $s_{\varpi,k}$ at a point p_i can be determined by adjusting the coefficient w_i. We might try to do this at each point p_i. If we just greedily do this, our estimates will fail due to the number of perturbations that we make growing with the number of points p_i used to construct the section $s_{\varpi,k}$. In general, the number of points M grows as k^{2n}. We can remove this difficulty by refining our lemma on choosing centers of points:

Proposition 3.5 Let $D > 0$. There exists a number $N(D) = O(D^{2n})$ independent of k for which the centers $\{p_i\}_{i \in I}$ can be chosen to satisfy the following partitioning condition. There is a partition $I = I_1 \cup \cdots I_{N(D)}$ so that points in I_k are at least D from each other.

Our strategy for perturbations will be to now perturb all of the points in I_k simultaneously. We will construct a series of coefficients ϖ_α such that over the balls centered on $I_0 \cup, \ldots I_\alpha$, the section $s_{\varpi_\alpha,k}$ is at least η_α transverse for some constant η_α not dependent on k. Suppose the section $s_{\varpi_{\alpha-1},k}$ is $\delta_{\alpha-1}$ transverse on the balls centered on $I_0 \cup, \ldots I_{\alpha-1}$. As the sections s_k have a nice exponential drop off, we know the following:

Lemma 3.6 Let $\varpi_{\alpha-1}$ and ϖ_α differ only in coefficients I_α, and there only by an amount δ. Then the C^1 norm of $s_{\varpi_{\alpha-1},k} - s_{\varpi_{\alpha-1},k}$ restricted to any ball is $O(\delta)$.

Since η-transversality is C^1 norm stable,

Corollary 3.7 Let $\varpi_{\alpha-1}$ and ϖ_α differ only in coefficients I_α, and there only by an amount δ. Suppose that $s_{\varpi_{\alpha-1},k}$ is η_α transverse to 0 on the balls centered over $I_0 \cup, \ldots I_\alpha$. Then $s_{\varpi_\alpha,k}$ is $\eta_\alpha - O(\delta)$ transverse to the zero section on the same balls.

So, by letting δ be $\eta/(2C)$, we can safely perturb our sections and preserve η transversality. We now use the distance between the centers of each I_α to show that we can simultaneously perturb these sections. The required condition that simultaneous perturbations don't effect each other very much is

$$\exp(-D^2/5) \leq \frac{1}{2} Q_p(\delta_\alpha)$$

In this case, the new section will be $\eta_\alpha = \frac{1}{2} Q_p(\delta_\alpha)\alpha$ transverse to the zero section on the new balls. In order to keep the old estimates intact, we need $\delta Q_p(\delta) \leq \eta_{\alpha-1}$. This gives us the required relation

$$\eta_\alpha = Q_p(\eta_{\alpha-1})\eta_{\alpha-1}$$

This inductive relation eventually expands out to the bound

$$Q_p(\eta_{\alpha-1}) \geq \frac{C}{(\alpha \log \alpha)^p}$$

for some constant C. Letting α take its maximal value

$$\begin{aligned} Q_p(\eta_{\alpha-1}) &\geq \frac{C}{(N \log N)^p} \\ &\geq \frac{C}{D^{2np+1}} \end{aligned}$$

For large enough D, we have that

$$\geq e^{-D^2/5}$$

which is the required estimate so that the balls do not effect each other a lot. This leaves the proof of the existence of global sections down to the proof of the main technical bound.

4 Proof of the Main Technical Bound

For this section we follow Auroux's Paper. Restated:

Theorem 4.1 *Main Technical Bound*

Let B^+ be the ball of radius $\frac{3}{2}$ in $\mathbb{C}^n$, and let $f : B^+ \to C$ be a function. Assume that f satisfies the bounds

$$|f| \leq 1 \quad |\bar{\partial} f| \leq \eta$$

Then there exists a $w \in \mathbb{C}$ with $|w| \leq \delta < 1/4$ such that $f - w$ is η-transverse to 0 over the interior ball B of radius 1. The quantity η is dependent on δ as

$$\eta = \delta Q_p(\delta)$$

and $Q_p(\delta) = \log^{-1}(\delta)^{-p}$, where p depends only on m and n.

While we use the case where $m = 1$, this bound turns out to be very difficult to prove. Auroux observes that in this particular case we can work instead with the case $m > n$ by perturbing sections in the jet bundle of $L^{\otimes k}$ instead.

Theorem 4.2 *Weakened Techinical Bound*

Let B^+ be the ball of radius $\frac{3}{2}$ and let $f : B^+ \to \mathbb{C}$ be a function. Assume that f satisfies the bounds

$$|f| \leq 1 \quad |\bar{\partial} f| \leq \eta$$

Then there exists a $w \in\in \mathbb{C}^m$ with $|w| \leq \delta < 1/4$ such that $f - w_0 - \sum w_i z_i$ is η transverse to 0 over the interior ball B of radius 1.

This turns out to be sufficient for us– the asymptotic drop off of the functions ensures that it does not matter if we work with perturbations that are constants, or perturbations that are linear. This technical bound depends on the following lemma:

Lemma 4.3

Let B^+ be the ball of radius $\frac{3}{2}$ in $\mathbb{C}^n$, and let $f : B^+ \to \mathbb{C}^m$ be a function with $m > n$. Assume

$$|f| \leq 1 \quad |\bar{\partial} f| \leq \eta$$

Then there exists a $w \in \mathbb{C}^m$ with $|w| \leq \delta < 1/4$ such that $|f - w| \geq \eta$ over the interior ball.

This is even better than η-transverse– this is η avoidance.

Proof. Outline of the proof:

1. Without loss of generality , assume that $m = n + 1$. Now we approximate f by a polynomial. The degree of this polynomial is $d = O(\log(\eta^{-1})$ and the polynomial is chosen so that $|f - g| \leq c\eta$.
2. If we can find $w \in \mathbb{C}^{n+1}$ with $|w| \leq \delta$, and $|g - w| \geq (c + 1)\eta$ over the ball B, then we get that $|f - w| \geq \eta$ everywhere. How do we find such a w for this polynomial?
3. Notice g is contained in an algebraic hypersurface of degree at most $(n + 1)d^n$.
4. Bound the volume of intersection of balls with H.

5. Use this to create a neighborhood of the hypersurface H, and therefore a neighborhood of C. Use the bound on the volume of the hypersurface to produce a point of sufficient distance from H.

■

Proposition 4.4 We can approximate f by a polynomial of degree $d\log(\eta^{-1})$.

Proof. Let $f = (f_0, \dots f_m)$. For each of the f_i, first approximate f_i by a holomorphic $\tilde{f}_i$ on the ball of radius 1.

Claim 4.5 There exists constant K such that any smooth $f : B^+ \to \mathbb{C}$ there is a holomorphic function $\tilde{f}$ defined on the interior B so that

$$\|f - \tilde{f}\|_1 \leq K(|\bar{\partial} f\|_0 + \|\nabla\bar{\partial} f\|_0).$$

You can solve this by finding the solution to the $\bar{\partial}$ problem

$$\|T(\bar{\partial} f)\|_{L^2(r'\Delta^+)} \leq C\|\bar{\partial} f\|_{L^2(\Delta^+)}.$$

Then $\tilde{f} = f - T(p)$ is holomorphic. Let $h = \tilde{f} - f$. The L^2 norm of h and C^1 norm of $\bar{\partial} h = \bar{\partial} f$ are B^+ are bounded by multiples of $\|\bar{\partial} f\|_1$. So is the C^1 norm of H.

Claim 4.6 Let $f : B^+ \to \mathbb{C}$ be a holomorphic function. Then for any $\epsilon < 1/2$, there is a complex polynomial g of degree less that $C\log\epsilon$ so that $|f(z) - g(z)|, |\partial f - \partial g| \leq \epsilon$ on B.

To prove this statement we use a truncation of a taylor series expansion of f. Calculate the coefficients in the taylor series by using the Cauchy integral formula. Then if g is truncated at degree ns, a bound on the Cauchy integral formula show sht at

$$|f(z) - g(z) \leq n2^{-s}$$

$$|\partial f - \partial g| \leq \sqrt{(}s + n)2^{-s}$$

which is enough to get our bound on the degree by

$$d \leq \log(-\epsilon)$$

■

Proposition 4.7 A polynomial $g : \mathbb{C}^n \to \mathbb{C}^m$ has image contained in an algebraic hypersurface degree at most $(n+1)d^n$.

Proof. If not, every nonzero polynomial of degree at most D in $n+1$ variables is non-identically zero of degree at most dD in n variables.

- The space of polynomials of degree at most D in $n+1$ variables is $\binom{D+n+1}{n+1}$.
- The space of polynomials of degree at most dD in n variables is $\binom{dD+n}{n}$.

If $D = (n+1)d^n$, then we have a lack of injectivity between these two sets of polynomials. ■

Proposition 4.8 Let $H \subset \mathbb{C}^{n+1}$ be a complex algebraic hypersurface of degree D. Then given any $r > 0$, and any $x \in \mathbb{C}^{n+1}$, the $2n$ dimensional volume of $H \cap B(x, r)$ is at most $DV_0 r^{2n}$, where V_0 is the volume of the unit ball of dimension $2n$. Moreover, if $x \in H$, then one also has $\mathrm{vol}_{2n} H(\cap B(c, r)) \geq V_0 r^{2n}$.

Let $\hat{B}$ be a ball of radius δ which satisfies $\eta = \delta\log(\delta^{-1})^{-p}$. Such a ball intersects H with volume at most $(n+1)V_0 d^n \delta^{2n}$. Cover $\hat{B}$ with balls of radius η so each point in $\hat{B}$ is only k redundantly covered. By the bound, one can check that the number of such balls required to cover $H \cap \hat{B}$ is $N = Cd^n\delta^{2n}\eta^{-2n}$.By taking δ to be larger than $C''d^{n/2}\eta$ for some fixed constant C'' dependent only on n, we will be able to find a point in $\hat{B}$ not contained in the cover of H, where the cover of H is of radius $(c+1)\eta$. Such a point has the property that $|f - w| \geq \eta$ every point in B.

5 Applications

Three additional applications are listed in Donaldson's paper. The first one is to provide some structure to sections found in the integrable case.

Proposition 5.1 Let (V, ω) be a Kähler manifold and $L \to V$ a hermitian holomorphic line bundle with curvature $-i\omega$. Then there is a constant $\eta > 0$ so that L^k has a holomorphic section with $|s| \leq 1$ and $|\partial s| \geq \eta\sqrt{k}$ everywhere.

This means that we can find transverse sections of line bundles in the case of the Kodaira embedding theorem. This gives us interesting bounds on divisors representing certain line bundles.

The second application is an adaptation of the Lefschetz Hyperplane theorem to symplectic manifolds.

Proposition 5.2 Let W_k be the zero set of asymptotically holomorphic sections arising from Donaldson's divisor theorem. Then when k is sufficiently large, the inclusion $W_k \to V$ induces an isomorphism on homotopy groups π_p for $p \leq n-2$, and a surjection on π_{n-1}.

The third application of Donaldson's theorem is a convergence result:

Proposition 5.3 Consider W_k as a current of degree 2. Then $W_k \to k\omega/2\pi$ as a current.