Some Practical Constructions with filtered A_∞ algebras

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March 15, 2019

Abstract

This paper reviews some basic definitions and notations for filtered (curved) A_∞ algebras. Much of the theory is presented using trees to diagrammatically express curved A_∞ relations, with particular attention spent to bounding cochains. In addition to providing a proof of the curved homological perturbation lemma, this exposition gives explicit chain models for mapping cones, fiber products, mapping cylinders, and homotopy squares. These tools are developed for the purpose of extending the statements (where possible) of [BC14] to the curved setting.

Contents

1 A refresher on curved A_∞ algebras

1.1 An A_∞ refresher

These notes are partly based on the already excellent exposition on non-curved A_∞ algebras [Kel99], and [Zha13] which explores deformation theory and curved A_∞ algebras in more detail, as well as [Fuk+00]. We will review curved A_∞ algebras, their morphisms and deformations. For reasons related to the convergence of the constructions of deformations (which will frequently involve infinite sums) we will work with the theory of *filtered A_∞ algebras.*

Definition 1.1.1 ([Fuk+00]). *Let R be a commutative ring with unit. The* universal Novikov ring *over R is the set of formal sums*

$$\Lambda_{\geq 0} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i\to\infty} \lambda_i = \infty. \right\}$$

Let k be a field. The Novikov Field *is the set of formal sums*

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} | \lambda \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \right\}$$

An energy filtration *on a graded Λ-module $A^\bullet$ is a filtration $F^{\lambda_i} A^k$ so that*

- *Each A^k is complete with respect to the filtration, and has a basis with valuation zero over Λ.*
- *Multiplication by T^λ increases the filtration by λ.*

Definition 1.1.2. *Let $A^\bullet$ be a graded Λ-module. A filtered A_∞ structure $(A^\bullet, m^k)$ is a graded Λ module $A^\bullet$ with Λ-linear cohomologically graded higher products for each $k \geq 0$*

$$m^k : (A^\bullet)^{\otimes k} \to (A^{\bullet+2-k})$$

satisfying the following properties:

- Energy Filtration: *The product respects the energy filtration in the sense that :*
$$m^k(F^{\lambda_1} A^\bullet, \cdots, F^{\lambda_k} A^\bullet) \subset F^{\sum_{i=1}^k \lambda_i} A^\bullet$$
- Non-Zero Energy Curvature: *The obstructing curvature term has positive energy,*
$$m^0 \in F^{\lambda > 0}(A^\bullet)$$
- Quadratic A_∞ relations *For each $k \geq 0$,*
$$\sum_{j_1+i+j_2=k} (-1)^{\clubsuit} m^{j_1+j_2+1}(\mathrm{id}^{\otimes j_1} \otimes m^i \otimes \mathrm{id}^{\otimes j_2}) = 0.$$
The value of $\clubsuit$ is determined on an input element $a_1 \otimes \cdots \otimes a_k$ by
$$\clubsuit = |a_{k-j_1}| + \cdots + |a_k| - i.$$

We say that $A^\bullet$ is unital *if there exists an element e_A such that*

$$m^{k_1+1+k_2}(\mathrm{id}^{\otimes k_1} \otimes e \otimes \mathrm{id}^{\otimes k_2}) = \begin{cases} \mathrm{id} & k_1 + k_2 = 1 \\ 0 & k_1 + k_2 \neq 1 \end{cases}.$$

For the purposes of exposition, we will ignore the sign $\clubsuit$ from here on own.

If $m^0 = 0$, then $(A^\bullet, m^1)$ is a chain complex, and we say that $A^\bullet$ is *uncurved* or *tautologically unobstructed*, otherwise, we say that $A^\bullet$ is curved. We from now on suppress the cohomological index, and when the product structure is clear, we will simply notate an A_∞ algebra by A.

Definition 1.1.3. *Let A be a filtered A_∞ algebra. An* ideal *of A is a subspace $I \subset A$ so that for every $i \in I$, and $a_1, \ldots a_{k-1} \in A$, we have that*

$$m^k(a_1 \otimes \cdots \otimes a_j \otimes i \otimes a_{j+1} \otimes \cdots a_{k-1}) \in I.$$

The quotient of a A_∞ algebra by an ideal is well defined. The filtration gives us a natural ideal of the A_∞ algebra.

Given (A, m^k) a filtered A_∞ algebra, define the positive filtration ideal $A_{>0} := \{a \in A \mid \mathrm{val}(a) > 0\}$. We then may recover a uncurved A_∞ algebra by taking the quotient,

$$A_{=0} := A/A_{>0}.$$

This is always an uncurved as the m^0 term is required to always have positive energy.

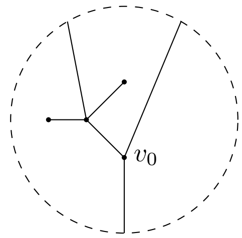


Figure 1: An example of a planar rooted tree with some semi-infinite leaves

1.2 From Trees to the Relations

The A_∞ relations are described by large compositions of multilinear maps, and it is frequently convenient to notate these large compositions of multilinear maps using the languages of trees.

Definition 1.2.1. *A* planer rooted tree with some semi-infinite leaves *is a tree T with the following additional data:*

- *An ordering of the leaves of T arising from a planar embedding of T.*
- *A choice of leaf e_0 called the* root *of T.*
- *A choice E^c of non-root leaves called the* semi-infinite leaves *or* external leaves.

When we say that v is a vertex of a planar rooted tree with semi-infinite leaves, we will always mean that v is a vertex of degree greater than 1, or a vertex of degree 1 which does not belong to a semi-infinite leaf or root edge.
If T is planar rooted tree with some semi-infinite leaves with at least 1 vertex, we denote by v_0 the vertex which is connected to the root edge.

One should imagine that a planar rooted tree is a rooted tree with an planar embedding into the disk with some subset of the leaf vertices on the boundary of the disk. From now on we will always use the word "tree" to describe a planar rooted tree with some semi-infinite edges. We define the *valence* of a tree T to be the number of external leaves, and write

$$\nu(T) := |E^c|.$$

The external leaf set E^c inherits an ordering $\{1, 2, \ldots, \nu(T)\}$ from the ordering of the leaves. Since T is a rooted tree, to each vertex we have an ordered upward edge set, $E_v^\uparrow$, and a downward edge $e_v^\downarrow$. Similarly, to each edge we have an upward vertex $v_e^\uparrow$ and downward vertex $v_e^\downarrow$.

Definition 1.2.2. *A* labelling *L of a tree T is an assignment to*

- *Each edge a vector space A_e.*
- *Each vertex a morphism*

$$f^v : \bigotimes_{e \in E_v^\uparrow} A_e \to A_{e_0}.$$

To each labelled tree (T, L), we obtain a morphism

$$f^{(T,L)} : \left(\bigotimes_{e \in E^c} A_e \right) \to A_{e_0}.$$

Notation 1.2.3. *In the event where there is a fixed algebra A so that for all $e \in E^c \cup e^0$, the algebras agree $A_e = A$, we will use the letter m to denote that this should be interpreted as a product relation on A,*

$$m^{(T,L)} : A^{\otimes \mathrm{val}(T)} \to A.$$

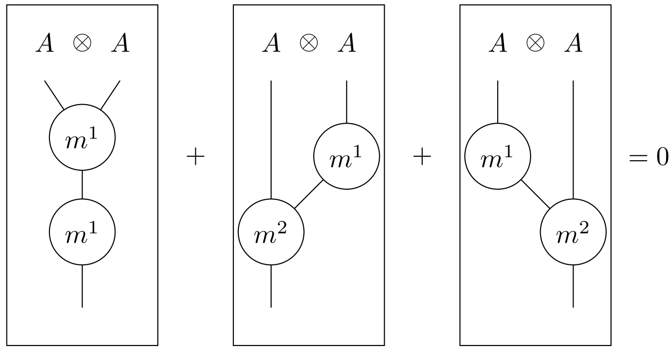


Figure 2: The $k = 2$ quadratic A_∞ relation expressed by a sum over trees.

Remark 1.2.4. *To specify the data of a labeled tree (T, L), it suffices to specify labels of compatible morphisms on the internal vertices, as one can recover the edge data from the domain and codomains of these morphisms.*

The quadratic A_∞ relations may be restated as the following sum over trees, which is also displayed in fig. 2

$$\sum_{\substack{(T,L)\ |\ \nu(T)=k \\ |V(T)|=2, L(v)=m^{\deg(v)-1}}} m^{(T,L)} = 0.$$

1.3 Morphisms of filtered A_∞-algebras

There is a well-defined notion of morphism between filtered A_∞ algebras.

Definition 1.3.1. *Let (A, m_A^k) and (B, m_B^k) be a pair of filtered- A_∞ algebras. A* weakly-filtered A_∞ homomorphism *from A to B is a sequence of graded maps*

$$f^k : A^{\otimes k} \to B$$

satisfying the following conditions:

- Weakly Filtered *The maps nearly preserve energy*

$$f^k(F^{\lambda_1}A, \cdots, F^{\lambda_k}A) \subset F^{-c\cdot k+\sum_{i=1}^k \lambda_i}B$$

for some fixed constant c called the energy loss of f *with $c < |m_A^0|$.*

- Quadratic A_∞ relations *The f^k, m_A^k and m_B^k mutually satisfy the quadratic curved A_∞ homomorphism relations*

$$\sum_{(j_1+i+j_2=k)} \pm f^{j_1+j_2+1}(\mathrm{id}^{\otimes j_1} \otimes m_A^i \otimes \mathrm{id}^{\otimes j_2}) = \sum_{i_1+\cdots+i_j=k} \pm m_B^j(f^{i_1} \otimes \cdots \otimes f^{i_j})$$

Suppose that A is an A_∞ algebra with unit. We say that f^k is a unital A_∞ *homomorphism if*

$$\begin{aligned} f^1(e_A) =& e_B \\ f^k(\mathrm{id}^{\otimes j_1} \otimes e_A \otimes \mathrm{id}^{\otimes j_2}) =& 0. \end{aligned}$$

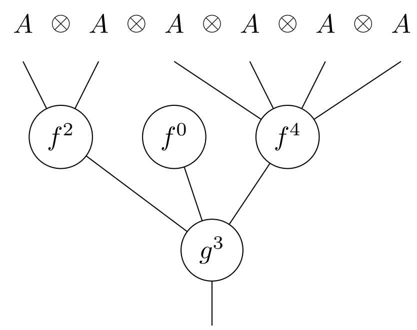


Figure 3: A typical term which appears in the composition $(g \circ f)^6$.

The quadratic A_∞ homomorphism relation may also be written as

$$\sum_{\substack{(T,L)\ \mid\ \nu(T)=k, |V^c|=2 \\ \text{Vertex above root labeled } f^i \\ \text{Other vertex labeled } m_A^j}} f^{(T,L)} = \sum_{\substack{(T,L)\ \mid\ \nu(T)=k, \\ \text{Vertex above root labeled } m_B^i \\ \text{every other vertex labeled } f^j}} f^{(T,L)}.$$

This can be re-expressed as:

$$\sum_{\substack{(T,L)\ \mid\ \nu(T)=A^{\otimes k} \\ \text{At most one vertex labeled } m_A^i \text{ or } m_B^i \\ \text{Every other vertex labeled } f^j}} f^{(T,L)} = 0.$$

Our tree notation becomes more useful for constructing new morphisms out of old.

Claim 1.3.2. *Let $f^{\otimes k} : A^{\otimes k} \to B$ and $g^{\otimes k} B^{\otimes k} \to C$ be two filtered A_∞ homomorphisms. Then*

$$(g \circ f)^k := \sum_{\substack{(T,L)\ \mid\ \nu(T)=k, \\ \textit{Vertex above root labeled } g \\ \textit{every other vertex labeled } f^j}} (g \circ f)^{(T,L)}.$$

is an A_∞ homomorphism.

See fig. 3 for a typical term which appears in the composition.

1.4 Deformations of A_∞ algebras

The presence of higher product structures gives us additional wiggle room to deform the product structures on filtered A_∞ algebra structure. We will be mainly interested when we can deform a given curved A_∞ algebra into an uncurved one. This is useful as the theory of uncurved A_∞ algebras is easier to work with than the theory of curved A_∞ algebras. In particular, a large portion of the theory can be reduced to algebra on the level of homology.

Notation 1.4.1. *As a shorthand, we write*

$$(\mathrm{id} + a)^{\binom{n+k}{n}_a} = \sum_{j_0+\cdots+j_k=n} (a^{\otimes j_0} \otimes \mathrm{id} \otimes a^{\otimes j_1} \otimes \mathrm{id} \otimes \cdots \otimes a^{\otimes j_{k-1}} \otimes \mathrm{id} \otimes a^{\otimes j_k}).$$

for the sum over all monomials containing $n + k$ terms, n of which are a.

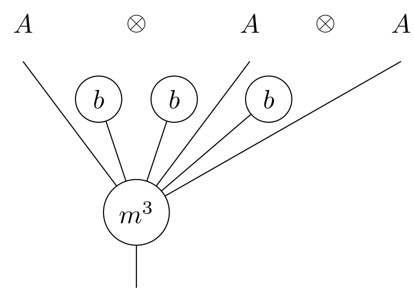


Figure 4: A typical tree contributing to m_b^3.

Definition 1.4.2. *Let $a \in A$ be an element of positive Novikov valuation. The a-*deformed product structure *on A is the product*

$$m_a^k := \sum_n \sum_{j_0 + \cdots j_k = n} m^{k+n} \left((\mathrm{id} + a)^{\binom{n+k}{n}_a} \right).$$

We call this a graded deformation *if the element a has homological degree 1.*

In the language of trees the deformed product is

$$m_a^k = \sum_{\substack{(T,L) \mid \nu(T) = k \\ T \text{ has unique non-leaf vertex labeled } m^n \\ \text{Every internal leaf is labelled } a}} m^{(T,L)}.$$

See fig. 4 for one of the terms of this sum. Note that this will be an infinite sum, as the number of trees with a bounded number of external leaves need not be bounded in the number of internal leaves. However, each internal leaf contributes some valuation to the composition, so at a bounded valuation the number of trees contributing to m_a^k is finite. The ensures convergences.

Claim 1.4.3. *(A, m_a^k) is again a filtered curved A_∞ algebra.*

Example 1.4.4. *The simplest of a deformation is in a DGA where $m^i = 0$ for $i \neq \{1, 2\}$. In this case, the deformed product becomes*

$$d_a^1(x) = d(x) + 2\,(a \wedge x)$$

which is the standard twisting of the differential on a differential graded algebra.

We are interested in the cases where (A, m_a) gives us a well defined homology theory even though A itself may be curved.

Definition 1.4.5. *We say that $a \in A$ is a* bounding cochain *or* Maurer-Cartan solution *if*

$$m_a^0 = 0.$$

Suppose that A has a unit. We say that $a \in A$ is a weak bounding cochain *or* weak Maurer-Cartan solution *if*

$$m_a^0 = W \cdot e_A,$$

where e_A is a unit, and W is some constant called the obstruction superpotential.

The presence of either a bounding cochain or weak bounding cochain is enough to give us a well defined homology theory. In the weak bounding case, we have that

$$m_a^1 \circ m_a^1 = m^2(\mathrm{id} \otimes (W \cdot e)) - m^2((W \cdot e) \otimes \mathrm{id}) = 0.$$

The Maurer-Cartan equation is sometimes written as

$$m_a^0 = \sum_n m^n(a) = e^a.$$

Definition 1.4.6. *Let A be an A_∞ algebra. The* space of Maurer-Cartan elements *is defined as*

$$\mathcal{MC}(A) := \{a \in A \mid m_a^0 = 0\}.$$

We say that A is unobstructed *if this space is non-empty.*

The space of weak Maurer-Cartan elements *is the space*

$$\mathcal{MC}_W(A) := \{a \in A \mid m_a^0 = W \cdot e\}$$

and we say that A is weakly unobstructed *if this space is non-empty.*

The Maurer-Cartan equation is non-linear. In the event that the Maurer-Cartan space contains a linear subspace, then 0 is a Maurer-Cartan element, and the algebra A is uncurved.

Lemma 1.4.7. *Let $f : A \to B$ be a weakly filtered A_∞ morphism (preserving units) of energy loss c. Then there exists a* pushforward *map between the (weak) bounding cochains on A of valuation greater than c, and the weak bounding cochains of B given by*

$$\begin{aligned} f_* : \mathcal{MC}_W(A) \to& \mathcal{MC}_W(B) \\ b_a \mapsto& \sum_k f^k(b_a^{\otimes k}) \end{aligned}$$

Proof. In order for $\sum_k f^k(b_a^{\otimes k})$ to converge, it suffices for the energy of b_a to be greater than the energy loss of f, which was assumed. We want to show that $b_B = \sum_k f^k(b_a^{\otimes k})$ satisfies the (weak) Maurer-Cartan equation

$$\begin{aligned} \sum_k m_B^k(b_B^{\otimes k}) =& \sum_k m_B^k\left(\left(\sum_{j_1} f^{j_1}(b_A^{\otimes j_1})\right) \otimes \cdots \otimes \left(\sum_{j_k} f^{j_k}(b_A^{\otimes j_k})\right)\right) \\ =& \sum_k \sum_{j_1,\ldots,j_k} m_B^k(f^{j_1}(b_A^{\otimes j_1}) \otimes \cdots \otimes f^{j_k}(b_A^{\otimes j_k}) \\ =& \sum_l \sum_{j_1+j_2+\ldots,j_k=l} m_B^k((f^{j_1} \otimes \cdots \otimes f^{j_k}) \circ (b_A^{\otimes l}) \\ =& \sum_l \sum_{i_1+j+i_2=l} f^{i_1+i_2+1}(\mathrm{id}^{\otimes i_1} \otimes m_A^j \otimes \mathrm{id}^{\otimes i_2}) \circ (b_A^{\otimes l}) \\ =& \sum_{i_1,i_2} f^{i_1+i_2+1}\left(\mathrm{id}^{\otimes i_1} \otimes \left(\sum_j m_A^j(b_A^{\otimes j})\right), \mathrm{id}^{\otimes i_2}\right) \circ (b_A)^{\otimes(i_1+i_2)} \\ =& \sum_{i_1,i_2} f^{i_1+i_2+1}\left(b_A^{\otimes i_1} \otimes W \cdot e_A, b_A^{\otimes i_2}\right) \\ =& W \cdot e_B + W \cdot \sum_{i_1+i_2>0} f^{i_1+i_2+1,}\left(b_A^{\otimes i_1} \otimes e_A \otimes b_A^{\otimes} i_2\right) \end{aligned}$$

In the case where b_A is a bounding cochain, $W = 0$ and we are finished. In the case where $W \neq 0$, the fact f was required to be unital means that the right terms vanish. $\square$

Surprisingly, deformations commute with each other in the following sense:

Claim 1.4.8. *Let a_1, a_2 be elements of A. Then $(A^\bullet, (m_{a_1})_{a_2}) = (A, m_{a_1+a_2})$.*

Proof. A calculation shows that

$$\begin{aligned}
(m^k_{a_1})_{a_2} &= \sum_n m^{k+n}_{a_1}(\mathrm{id}\oplus a_2)^{\binom{n+k}{n}_{a_2}} \\
&= \sum_m \sum_n m^{k+m+n}(\mathrm{id}\oplus a_1)^{\binom{n+k+m}{m}_{a_1}} \circ (\mathrm{id}\oplus a_2)^{\binom{n+k}{n}_{a_2}} \\
&= \sum_{m+n} m^{k+m+n}(\mathrm{id}+a_1+a_2)^{\binom{n+m+k}{m+n}_{a_1+a_2}} = \qquad m^k_{a_1+a_2}
\end{aligned}$$

□

Remark 1.4.9. *Because the space of Maurer-Cartan elements is cut out by a non-linear equation it is unlikely that if a_0 and a_1 are bounding cochains that $a_0 + a_1$ is similarly a bounding cochain.*

Claim 1.4.10. *Let A and B be two filtered A_∞ algebras, and let $f : A \to B$ be a filtered A_∞ algebra morphism. Then there exists an A_∞ homomorphism*

$$f_\flat : (A, m_A) \to (B, (m_B)_{f_*(0)})$$

where $f_\flat$ is defined [1] *by*

$$f^k_\flat = \begin{cases} f^k & \textit{for } k > 0 \\ 0 & \textit{if } k = 0 \end{cases}$$

Claim 1.4.11. *Let A and B be two A_∞ algebras, and let $f : A \to B$ be a filtered A_∞ algebra morphism. Let $a \in A$ be a deforming element. Then there the map*

$$f_a : (A, (m^k_A)_a) \to (B, m^k_B).$$

Proof. Define f^k_a to be the map

$$f^k_a := \sum_n f^{k+n}(\mathrm{id}+a)^{\binom{n+k}{n}_a}.$$

We show that this satisfies the quadratic A_∞ relations by explicit computation.

$$\begin{aligned}
&\sum_{j_1+i+j_2=k} \pm f^{j_1+j_2+1}_a(\mathrm{id}^{\otimes j_1} \otimes m^i_{A,a} \otimes \mathrm{id}^{\otimes j_2}) \\
&= \sum_{j_1+i+j_2=k} \sum_{n_1,m,n_2} \pm f^{j_1+n_1+1+j_2+n_2}\left(\left((\mathrm{id}+a)^{\binom{j_1+n_1}{n_1}_a}\right) \otimes m^{i+m}\left((\mathrm{id}+a)^{\binom{i+m}{m}_a}\right) \otimes (\mathrm{id}+a)^{\binom{j_2+n_2}{n_2}_a}\right) \\
&= \sum_n \sum_{n_1+m+n_2=n} \sum_{j_1+i+j_2=k} \left(f^{k+n}(\mathrm{id}^{\otimes(j_1+n_1)} \otimes m^{i+m} \otimes \mathrm{id}^{\otimes(j_2+n_2)})\right) \circ (\mathrm{id}+a)^{\binom{k+n}{n}_a} \\
&= \sum_n \sum_{i_1+\cdots+i_h=n+k} m^h_B(f^{i_1} \otimes \cdots \otimes f^{i_h}) \circ (\mathrm{id}+a)^{\binom{k+n}{n}_a} \\
&= \sum_{l_1+\cdots+l_h=k} m^j_B(f^{l_1}_a \otimes \cdots \otimes f^{l_j}_a)
\end{aligned}$$

□

[1] The the notation is read "f-flat"', as this is the flat version of the curved A_∞ homomorphism f.

One may use the previous two claims to construct the pushforward map on bounding cochains, as

$$f_*(b) = (f_b)_*(0)$$

This, along with the statement on the pushforward of a bounding cochain, proves the following characterization of unobstructed A_∞ algebras.

Corollary 1.4.12. *Let A, B be two filtered A_∞ algebra. A zero-morphism $0 : A \to B$ with $f^i = 0$ for $i \geq 1$ exists if and only if B is unobstructed.*

2 Curved Homological Perturbation Lemma

2.1 A curved Homological Perturbation Lemma

In this section we prove a curved homological perturbation lemma.

Theorem 2.1.1 (Curved Homological Perturbation Lemma)**.** *Let B be a filtered A_∞ algebra, and $(A_{=0}, \mu^1_{A_{=0}})$ be a chain complex. Suppose there exist chain maps $\pi : B_{=0} \to A_{=0}$ and $i : A_{=0} \to B_{=0}$ so that*

- *There exists a weakly filtered chain homotopy $h : B \to B$ so that*

$$h \circ \mu^1_B + \mu^1_B = \mathrm{id} - i_0 \circ \pi$$

Then we can extend the chain structure on $A_{=0}$ to a filtered A_∞ structure (A, m^k_A), where the Λ-graded portion of m^1_A matches $\mu^1_{A_{=0}}$. For this choice of filtered A_∞ structure, the map π is a homotopy equivalence of filtered A_∞ algebras with explicit weakly filtered A_∞ homotopy inverse

$$i^k : A^{\otimes k} \to B.$$

If A already had an A_∞ structure so that π is a filtered A_∞ map, then the extended A_∞ structure on A can be chosen to match the original structure.

In the setting of non-curved A_∞ algebras, this statement exactly matches the usual statement of a curved A_∞ algebra. If B has no curvature, then the constructed A_∞ structure m^1_A matches d_A.

The remainder of this section is devoted to the proof of theorem 2.1.1.

We want to describe a sequence of maps $i^k : A^{\otimes k} \to B$ satisfying the A_∞ relations. The maps can be constructed inductively (see [Sei08]) but we will describe them using trees. To each tree with labellings we will associate a morphism from $i^T : A^{\otimes \nu(T)} \to B$ by taking a composition of morphisms specified by the adjacency data of the tree.

We now specify labellings (T, L^m_{hpl}) which will determine the product structure on A.

- We label each external leaf and root with the vector space A. We label each internal edge with the vector space B.
- If v is a vertex of T we label it with the morphism $h \circ m^{\deg(v)-1}$.
- If v is a vertex of T incident to an external leaf, we pre-compose with the appropriate tensor product of inclusions $i : A \to B$ and $\mathrm{id} : B \to B$ so that the domain of the label of v matches its upward edges.
- We post-compose at the vertex v_0 with the morphism $\pi : B \to A$.

We define the product structure on A by the sum over all stable trees of valency k as well

$$m^k_A := \sum_{\substack{T \mid \nu(T)=k \\ \deg(v)\neq 2}} m^{(T, L^m_{hpl})}$$

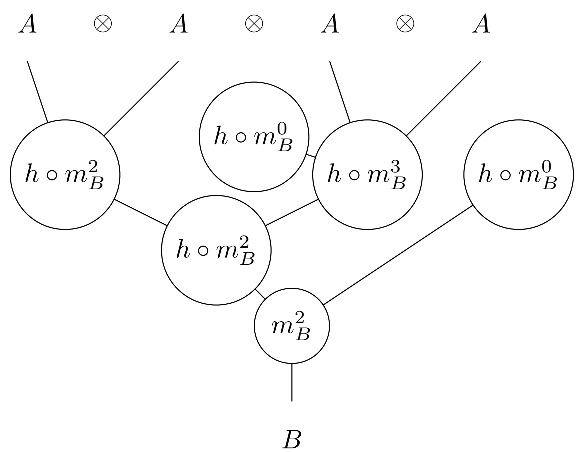


Figure 5: A typical example of a tree with a homological perturbation lemma labelling contributing to i.

To each tree T, we associate the homological perturbation lemma labelling (T, L^i_{hpl}), which is drawn in fig. 5.

- We label each external leaf with A. We label each internal edge and root with the label B.
- If v is an internal vertex of T, we label it with the morphism $h \circ m^{\deg(v)-1}$.
- If v is a vertex of T which is incident to an external leaf, we pre-compose with the appropriate tensor product of inclusions $i : A \to B$ and $\mathrm{id} : B \to B$ so that the domain of the label of v matches its upward edges.

We define maps i^k to be the sum of all such maps over stable trees of valency k,

$$i^k := \sum_{\substack{T \mid \nu(T)=k \\ \deg(v)\neq 2}} i^{(T, L^i_{hpl})}$$

Note that we have the following relations between the product structure and the constructed maps:

$$\pi \circ i^{(T, L^i_{hpl})} = m^{(T, L^m_{hpl})}.$$

Both i^k and m^k_A are defined by infinite sums. This sum converges over Λ, as the valuation of a morphism can be bounded below by the number of internal leaves when h is weakly filtered. At each valuation λ, there are most $\frac{\lambda}{\nu(h \circ m^0)}$ internal leaves in each $i^{(T,L)}$ contributing to i^k below that valuation. Because the number of stable trees with a fixed number of leaves is finite, we have the sum over all $i^{(T, L^i_{hpl})}$ of bounded valuation and fixed valency is bounded. This is sufficient to ensure convergence in the Novikov field.

We prove that the morphisms i^k and m^k mutually satisfy the quadratic A_∞ homomorphism relations. We omit the proof that the m^k_A satisfy the A_∞ relations as the proof is similar, but simpler.

Definition 2.1.2. *We say that T is a 1-unstable tree if there it has a single vertex v of degree 2. In this case, we call this vertex v the unstable vertex of T. Let T be a 1-unstable tree. The instability distance of T is the distance from the unstable vertex to v_0.*

If T is an unstable tree, we show that the quadratic A_∞ relations allow us to reexpress each $i^{(T,L^i_{hpl})}$ as a sum of unstable trees with greater instability distance. This process will eventually give us the full A_∞ relations for homomorphisms. We consider the following special labellings of trees.

- $i \circ \pi$ broken Trees. At the specified vertex v, we use the label $i \circ \pi \circ m_B^{\deg(v)-1}$ instead of the standard label. We call the corresponding labelling $L_v^{\pi\circ i}$.

  The $i \circ \pi$ broken trees can be also expressed in the following way. Let $T_v^\uparrow$ be the tree which is obtained by taking all edges upwards of v and the edge $e^\downarrow v$. v is the interior root vertex of the tree $T_v^\uparrow$. Let $T_v^\downarrow$ be the tree which consists of all edge not upward of v, so that v is an external leaf of this new tree. Then this $i \circ \pi$ broken tree is equivalent to the composition

  $$i^{(T,L_v^{i\circ\pi})} = i^{(T_v^\downarrow, L^i_{hpl})} \circ \mathrm{id}^{\otimes k^1} \otimes m^{(T_v^\uparrow, L^m_{hpl})} \otimes id^{\otimes k_1}.$$

  where k_1 is the number of leaves "left" of the vertex v, and k_2 is the number of leaves right of v, so that $\mathrm{val}(T) = k_1 + \mathrm{val}(T_v^\uparrow) + k_2$.

- id broken trees. At a specified vertex v, we choose the label $\mathrm{id} \circ m_B^{\deg(v)-1}$. We call the corresponding label L_v^{id}.

The following observations become the framework for proving the homological perturbation lemma

1. The sum over all id-broken stable trees at v_0 of fixed valence nearly gives the right hand side of the A_∞ relations.

$$\sum_{T \mid \mathrm{val}(T)=k} i^{(T,L^{\mathrm{id}}_{v_0})} = \sum_{i_1+\cdots+i_j=k, k>1} \pm m_B^j(i^{i_1} \otimes \cdots \otimes i^{i_j}) \tag{1}$$

2. The sum of all $i \circ \pi$ broken trees of fixed valence gives a large portion of the A_∞ relations,

$$\sum_{v\in T \mid \mathrm{val}(T)=k} i^{(T,L_v^{i\circ\pi})} = \sum_{\substack{(j_1+i+j_2=k)\\ i>1}} \pm i^{j_1+j_2+1}(\mathrm{id}^{\otimes j_1} \otimes m_A^i \otimes \mathrm{id}^{\otimes j_2}) \tag{2}$$

Let T be a tree. The subdivision tree T is tree $T \div e$ obtained by replacing the edge e with two edges, and a new vertex v_e. The subdivision of a tree is never stable, as the new vertex v_e has degree 2.

Claim 2.1.3 (Homotopy Identity)**.** *Let T be a tree. Let e be an interior edge with upward vertex v.*

$$i^{\left(T\div e, L^{\mathrm{id}}_{e_v}\right)} + i^{\left(T\div e, L^{\mathrm{id}}_{v_e^\uparrow}\right)} + i^{\left(T, L^{i\circ\pi}_{v_e^\uparrow}\right)} + i^{\left(T, L^{\mathrm{id}}_{v_e^\uparrow}\right)} = 0$$

Given a stable tree T and a vertex $v \in T$, the *expansions of T at v* are the planar trees T' with two vertices $v_\downarrow, v_\uparrow \subset V(T')$ so that the contraction $T'/\{v_\downarrow v_\uparrow\}$ is T, and under this contraction both $v_\downarrow$ and $v_\uparrow$ are identified with v.[2] If T' is an expansion of T at a vertex v, we label it $(T', L^{\mathrm{id}}_{v_\uparrow})$.

Claim 2.1.4 (Associativity Identity)**.** *Let T be a tree. Let $v \in T$ be any vertex. Then*

$$\sum_{\substack{(T', L^{\mathrm{id}}_{v_\uparrow})\\ T' \text{ an expansion of } T \text{ at } v}} i^{\left(T', L^{\mathrm{id}}_{v_\uparrow}\right)} = 0.$$

[2] The planarity condition is important here. For example, a trivalent vertex has 6 expansions where 3 of the expansions are isomorphic as trees but not as planar trees.

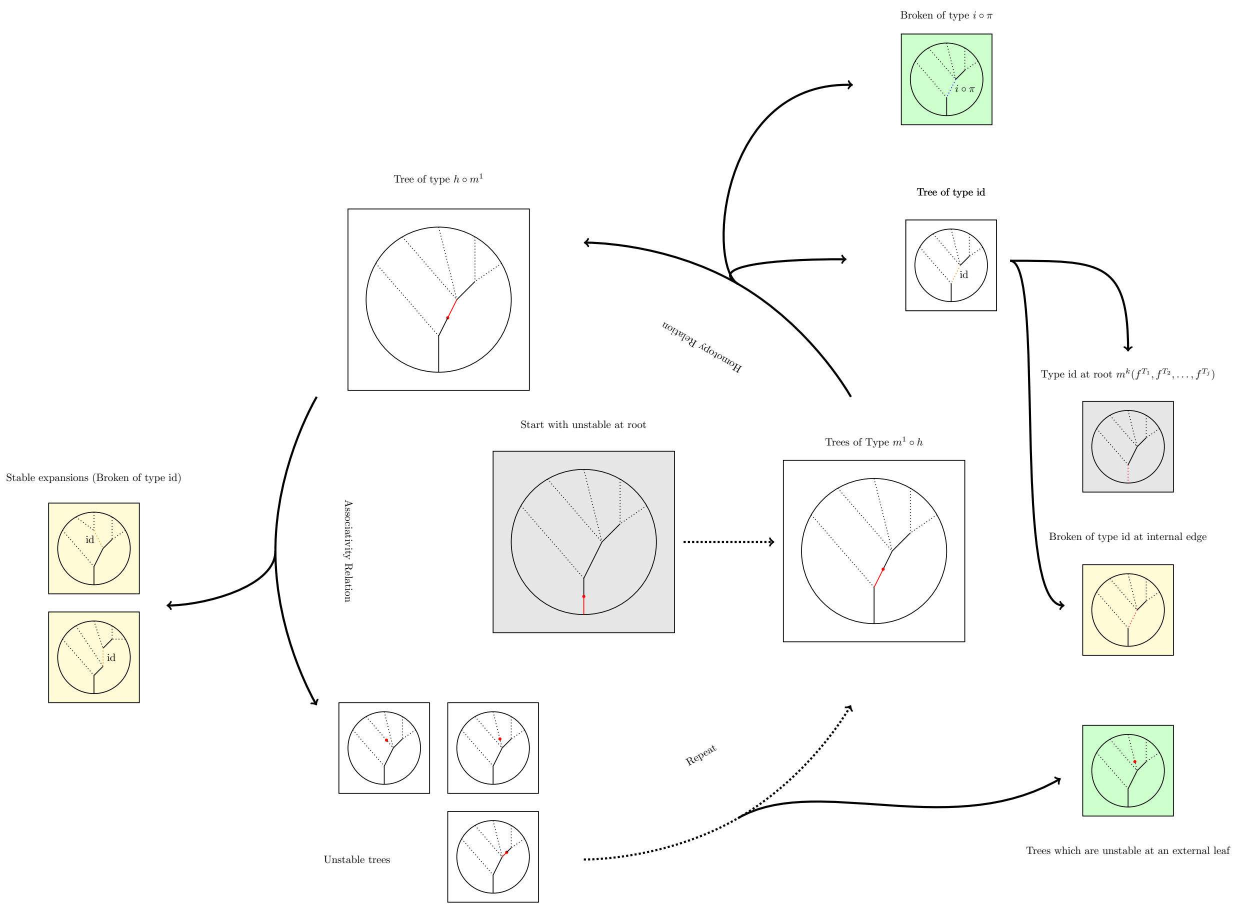


Figure 6: Repeated applications of homotopy and associative relations.

Lemma 2.1.5. *Let T be a stable tree.*

$$i^{(T\div e_0, L^{\mathrm{id}}_{v_{e_0}})} = \sum_{\substack{(T', L^{\mathrm{id}}_{v_\uparrow}) \\ T' \text{ a stable expansion of } T}} i^{\left(T', L^{\mathrm{id}}_{v_\uparrow}\right)} + \underbrace{\sum_{v \in V(T)} i^{(T, L^{\mathrm{id}}_v)}}_{\mathrm{id} \text{ broken trees}} + \underbrace{\sum_{v \in V(T)} i^{(T, L^{i\circ\pi}_v)}}_{i\circ\pi \text{ broken trees}} + \sum_{e \in E_c} i^{(T\div e, L^{\mathrm{id}}_{v_e})}.$$

Proof. Let T be a stable tree, and e a vertex in T. The tree $T\div e$ is a 1-unstable tree. We show that if e is not a external leaf, $i^{(T\div e, L^{id}_{e_v})}$ can be re-expressed as a sum of expansions of T, broken trees, and $i^{(T\div e', L^{id}_{e'_v})}$ where the edges e' have a greater distance from the root.

1. **Homotopy Step** Let $v_\uparrow$ be the upper vertex of the edge e. By claim 2.1.3,
$$i^{(T\div e, L^{id}_{e_v})} = i^{(T\div e, L^{\mathrm{id}}_{v_\uparrow})} + i^{(T, L^{i\circ\pi}_v)} + i^{(T, L^{\mathrm{id}}_v)}$$
The two terms on the right are broken trees, which are allowed terms in the expansion.
2. **Associativity Step** Let $E_{v_\uparrow}$ be the upward edge set of $v_\uparrow$. By claim 2.1.4
$$i^{(T\div e, L^{\mathrm{id}}_{v_\uparrow})} = \left(\sum_{\substack{(T', L^{\mathrm{id}}_{v_\uparrow}) \\ T' \text{ a stable expansion of } T \text{ at } v_\uparrow}} i^{(T', L^{\mathrm{id}}_{v_\uparrow})} \right) + \left(\sum_{\substack{(T\div e', L^{\mathrm{id}}_{v_{e'}}) \\ e' \in E_{v_\uparrow}}} i^{(T\div e', L^{\mathrm{id}}_{v_{e'}})} \right).$$

Each time we apply these two steps, we replace the 1-unstable tree $T \div e$ with stable broken trees, and 1-unstable trees of greater instability distance. The process terminates when unstable term has moved to the external leaves of T. □

We expand out this relation a little bit.

$$
\begin{aligned}
0 = {} & i^{(T\div e_0, L^{\mathrm{id}}_{v_{e_0}})} + i^{(T, L^{\mathrm{id}}_{v_0})} + \sum_{v\in V(T)} i^{(T, L^{i\circ\pi}_v)} + \sum_{e\in E_c} i^{(T\div e, L^{\mathrm{id}}_{v_e})} \\
& + \sum_{\substack{(T', L^{\mathrm{id}}_{v_\uparrow}) \\ T' \text{ a stable expansion of } T}} i^{(T', L^{\mathrm{id}}_{v_\uparrow})} + \sum_{v\in V(T), v\neq v_0} i^{(T, L^{\mathrm{id}}_v)} \\
= {} & \left(m^1_B \circ i^{(T,L)} + m_B^{\deg v_0 - 1} \circ \left(\bigotimes_{e\in E_{v_0}} i^{(T_{e_\uparrow}, L_{hpl})} \right) \right) + \left(\sum_{e\in E_c} i^{(T\div e, L^{\mathrm{id}}_{v_e})} + \sum_{v\in V(T)} i^{(T^\downarrow_v, L^i_{hpl})} \circ m^{(T^\uparrow_v, L^m_{hpl})} \right) \qquad (3) \\
& + \left(\sum_{\substack{(T', L^{\mathrm{id}}_{v_\uparrow}) \\ T' \text{ a stable expansion of } T}} i^{(T', L^{\mathrm{id}}_{v_\uparrow})} + \sum_{v\in V(T), v\neq v_0} i^{(T, L^{\mathrm{id}}_v)} \right).
\end{aligned}
$$

We now reduce the terms of eq. (3). From eq. (1), we conclude that

$$
\sum_{\substack{\nu(T)=k \\ T \text{ a stable tree}}} \left(m^1_B \circ i^{(T,L)} + m_B^{\deg v_0 - 1} \circ \left(\bigotimes_{e\in E_{v_0}} i^{(T_{e_\uparrow}, L_{hpl})} \right) \right) = \sum_{i_1+\cdots+i_j = k, k>1} \pm m^j_B (i^{i_1} \otimes \cdots \otimes i^{i_j}) \qquad (4)
$$

From eq. (2) we conclude that

$$
\sum_{\substack{\nu(T)=k \\ T \text{ a stable tree}}} \left(\sum_{e\in E_c} i^{(T\div e, L^{\mathrm{id}}_{v_e})} + \sum_{v\in V(T)} i^{(T^\downarrow_v, L^i_{hpl})} \circ m^{(T^\uparrow_v, L^m_{hpl})} \right) = \sum_{(j_1+i+j_2=k)} \pm i^{j_1+j_2+1} (\mathrm{id}^{\otimes j_1} \otimes m^i_A \otimes \mathrm{id}^{\otimes j_2}) \qquad (5)
$$

The main idea of 2.1.1 is to notice that the expansions and contractions of T will cancel out with expansions and contractions from other trees.

Proposition 2.1.6. *The sum over all trees of broken trees from expansion terms and broken trees from identity terms exactly cancel,*

$$
0 = \sum_{\substack{\nu(T)=k \\ T \text{ a stable tree}}} \left(\sum_{\substack{(T', L^{\mathrm{id}}_{v_\uparrow}) \\ T' \text{ a stable expansion of } T}} i^{(T', L^{\mathrm{id}}_{v_e^\uparrow})} + \sum_{v\in V(T), v\neq v_0} i^{(T, L^{\mathrm{id}}_v)} \right) \qquad (6)
$$

Proof. We first note that there is a bijection

$$
\begin{aligned}
\bigcup_{T' \text{ stable}, \nu(T')=k} \{e \text{ and internal edge of } T'\} \to & \bigcup_{T \text{ stable}, \nu(T')=k} \{ \text{ Stable expansions of } T\} \\
(T', e) \mapsto & (T', T/e)
\end{aligned}
$$

This, combined with the identification of internal edges of T' with non-root vertices gives us the equality

$$\sum_{\substack{\nu(T')=k \\ T' \text{ a stable tree}}} \sum_{v\in V(T), v\neq v_0} i^{(T,L_v^{\mathrm{id}})} = \sum_{\substack{\nu(T')=k \\ T' \text{ a stable tree} \\ e\in E^i(T')}} i^{(T,L^{\mathrm{id}}_{v_e^{\uparrow}})} = \sum_{\substack{\nu(T)=k \\ T \text{ is a stable tree} \\ T' \text{ is a stable expansion of } T}} i^{(T,L^{\mathrm{id}}_{v_e^{\uparrow}})}$$

proving the proposition. □

Remark 2.1.7. *In the non-curved setting, where $m^0 = 0$, there is a nice visualization of the above lemma. Consider the poset of metric trees with ordering given by the minor relation. This poset has a geometric realization as the standard cell decomposition of the Stasheff associahedra, and the above relation states that contraction labellings of a tree T are related to the trees in it's cellular boundary, while the expansion byproducts of a tree are related to the cells which it is a boundary of.*
In the curved setting, there is no nice geometric picture of the lemma. Instead, consider the poset of metric trees with internal leaves and a fixed number of external leaves, again with the minor ordering relation. This is an infinite poset with a unique maximal member. Again, the lemma above states that there is pairing between the expansion and contraction exhaustions of tree of fixed valence, by viewing them as the ends of edges in the Hasse diagram of this poset which are closer or farther from minimal element.

We are now in a place to prove the A_∞ relations. Taking the relation eq. (3) over all trees and applying eqs. (4) to (6)

$$0 = \sum_{\substack{\nu(T)=k \\ T \text{ is a stable tree}}} \left(\begin{array}{c} \underbrace{m_B^1 \circ i^{(T,L)} + m_B^{\deg v_0 - 1} \circ \left(\bigotimes_{e\in E_{v_0}} i^{(T_{e_\uparrow}, L_{hpl})} \right)}_{\text{eq. (4)}} \\ + \underbrace{\sum_{e\in E_c} i^{(T\div e, L^{\mathrm{id}}_{v_e})} + \sum_{v\in V(T)} i^{(T_v^{\downarrow}, L^i_{hpl})} \circ m^{(T_v^{\uparrow}, L^m_{hpl})}}_{\text{eq. (5)}} \\ + \underbrace{\sum_{\substack{(T', L^{\mathrm{id}}_{v_\uparrow}) \\ T' \text{ a stable expansion of } T}} i^{(T', L^{\mathrm{id}}_{v_\uparrow})} + \sum_{v\in V(T), v\neq v_0} i^{(T, L^{\mathrm{id}}_v)}}_{\text{eq. (6)}} \end{array} \right)$$

$$= \left(\sum_{i_1+\cdots+i_j=k} \pm m_B^j (i^{i_1} \otimes \cdots \otimes i^{i_j}) \right) + \left(\sum_{(j_1+i+j_2=k)} \pm i^{j_1+j_2+1} (\mathrm{id}^{\otimes j_1} \otimes m_A^i \otimes \mathrm{id}^{\otimes j_2}) \right) + 0$$

which, when rearranged, gives us the A_∞ relations.

2.2 Application: The replacement tool

We can use the curved homological perturbation lemma to prove a classic result from homological algebra.

Lemma 2.2.1 (Replacement Tool)**.** *Let A be a filtered A_∞ algebra, and let $B \subset A$ be a filtered A_∞ ideal, giving us the short exact sequence*

$$B \to A \to (A/B)$$

Let $f : B \to B'$ be an A_∞ homomorphism which is a homotopy equivalence on the chain level. Then there exists a filtered A_∞ algebra A' with A' homotopic to A and a short exact sequence

$$B' \to A' \to (A'/B')$$

with $(A'/B') = (A/B)$.

Proof. For convenience, we write $C = (A/B)$. We exhibit an A_∞ structure on $B' \oplus C$. Let $f : B \to B'$ be our prescribed A_∞ homotopy which is an equivalence. As a vector space, $A' = B' \oplus C$, and there exists a map $\pi : A \to A'$. Furthermore, there is an inclusion of chain complexes $i : A'_{=0} \to A_{=0}$ which is a homotopy inverse of the identity. We construct the A_∞ structure on A' using the homological perturbation lemma, along with the homotopy inverse map. □

The replacement tools allows us to modify filtered A_∞ algebras by identifying subalgebras and replacing them with homotopic subalgebras.

3 Cones and Fiber Products

In this section, we show that the classical constructions of mapping cones and fiber products extend to tautologically unobstructed A_∞ algebras and A_∞ algebras respectively.

3.1 Mapping Cones

We begin by describing the mapping cone construction for tautologically unobstructed A_∞ algebras.

Definition 3.1.1 (Left A_∞ module)**.** *Let A be a tautologically unobstructed A_∞ algebra. A* left module over A *is a graded Λ-module M, along with a sequence of maps*

$$m^{k-1|1}_{A|M} : A^{\otimes k-1} \otimes M \to M$$

satisfying the following quadratic A_∞ module relation:

$$\begin{aligned} 0 = &\sum_{j_1+j=k} m^{j_1|1}_{A|M}(\mathrm{id}_A^{\otimes j_1-1} \otimes m^{j-1|1}_{A|M}) \\ &+ \sum_{j_1+j+j_2=k \mid j_2\neq 0} m^{j_1+j_2|1}_{A|M}(\mathrm{id}_A^{\otimes j_1} \otimes m^j_A(a) \otimes \mathrm{id}_A^{\otimes j_2-1} \otimes \mathrm{id}_B). \end{aligned}$$

In these quadratic relations it appears that we are taking the sum over two different types of compositions. However, this can also be described as the sum over all trees with 2 internal vertices and leaves labelled $A^{\otimes k-1} \otimes M$, and internal vertices labelled $m^{j|1}_{A|M}$ or m^j_A.

Given a morphism of A_∞ algebras $f : A \to B$, there exists a change of base formula from $A - \mathbf{Mod}$ to $B - \mathbf{Mod}$.

Claim 3.1.2. *Suppose we have uncurved morphism of uncurved A_∞ algebras $f : A \to B$. Then the products*

$$\begin{aligned} m^{k|1}_f : A^{\otimes k} \otimes B \to &B \\ m^{k|1}_f = &\sum_{k=j_1\ldots j_i} m^{i+1}_B(f^{\otimes j_1} \otimes \cdots \otimes f^{j_i} \otimes \mathrm{id}_B). \end{aligned}$$

make B a A_∞ module over A.

Proof. We delay the proof of this statement until we prove the curved A_∞ bimodule relations in claim 3.2.3. □

We use the structure of this multiplication to construct mapping cones in the category of A_∞ algebras. Let $\pi^{i|j}_{A|B} : (A \oplus B[1])^{\otimes i+j} \to A^{\otimes i} \oplus B^{\otimes j}$ be the standard projection.

Definition 3.1.3. *Let $f : A \to B$ be a morphism of A_∞ algebras. The* cone of f *is the A_∞ algebra on the graded vector space*

$$\mathrm{cone}(f) = A \oplus B[1]$$

equipped with the higher product structures

$$m^k_{\mathrm{cone}} := \left(\left(m^k_A \circ \pi^{k|0}_{A|B}\right) \oplus \left(f^k \circ \pi^{k|0}_{A|B} + m^{k-1|1}_f \circ \pi^{k|1}_{A|B}\right)\right)$$

$$m^k_{\mathrm{cone}}\left(\bigotimes_{i=1}^k (a_i, b_i)\right) = \left(m^k_A\left(\bigotimes_{i=1}^k a_i\right), f^k\left(\bigotimes_{i=1}^k a_i\right) + m^{k-1|1}_f\left(\left(\bigotimes_{i=1}^{k-1} a_i\right) \otimes b_k\right)\right)$$

Proof. The proof is a verification of the A_∞ structure. It is immediate from the quadratic relations on A that the A-component of $\sum_{j_1+j+j_2=k} m^{j_1+j_2+1}_{\mathrm{cone}}(\mathrm{id}^{\otimes j_1} \otimes m^j_{\mathrm{cone}} \otimes \mathrm{id}^{j_2})$ will be zero. It therefore suffices to look at the B-component of this relation. We will use that

$$\pi^{j_1+1+j_2|0} \circ (\mathrm{id}^{\otimes j_1}_{\mathrm{cone}} \otimes m^j_{\mathrm{cone}} \otimes \mathrm{id}^{j_2}_{\mathrm{cone}}) = (\mathrm{id}^{\otimes j_1}_A \otimes m^j_A \otimes \mathrm{id}^{\otimes j_2}_A) \circ \pi^{k|0}_{A|B}.$$

$$\pi^{j_1+j_2|1} \circ (\mathrm{id}^{\otimes j_1}_{\mathrm{cone}} \otimes m^j_{\mathrm{cone}} \otimes \mathrm{id}^{j_2}_{\mathrm{cone}}) = \begin{cases} (\mathrm{id}^{\otimes j^1}_A\ m^k_A \otimes \mathrm{id}^{\otimes j_2 - 1}_A \otimes \mathrm{id}_B) \circ \pi^{k-1|1}_{A|B} & \text{if } j_2 \neq 0 \\ \mathrm{id}^{j_1}_A \otimes (m^{j-1|1}_f \circ \pi^{j-1|1}_{A|B} + f^j \circ \pi^{j|0}_{A|B} & \text{if } j_2 = 0 \end{cases}$$

The B-component of the quadratic A_∞ relations is

$$\begin{aligned}
&\pi^{0|1}_{A|B} \circ \sum_{j_1+j+j_2=k} m^{j_1+j_2+1}_{\mathrm{cone}}(\mathrm{id}^{\otimes j_1}_{\mathrm{cone}} \otimes m^j_{\mathrm{cone}} \otimes \mathrm{id}^{j_2}_{\mathrm{cone}}) \\
&= \sum_{j_1+j+j_2} \begin{pmatrix} f^{j_1+1+j_2}(\pi^{j_1+1+j_2|0}(\mathrm{id}^{\otimes j_1}_{\mathrm{cone}} \otimes m^j_{\mathrm{cone}} \otimes \mathrm{id}^{j_2}_{\mathrm{cone}}) \\ +m^{k-1|1} \circ \pi^{k|1}(\mathrm{id}^{\otimes j_1}_{\mathrm{cone}} \otimes m^j_{\mathrm{cone}} \otimes \mathrm{id}^{j_2}_{\mathrm{cone}}) \end{pmatrix} \\
&= \sum_{j_1+j+j_2} f^{j_1+1+j_2} \circ (\mathrm{id}^{\otimes j_1}_A \otimes m^j_A \otimes \mathrm{id}^{\otimes j_2}_A) \circ \pi^{k|0} \\
&\quad + \sum_{j_1+j+j_2|j_2\neq 0} m^{j_1+j_2|1}_f \circ \mathrm{id}^{\otimes j^1}_A\ m^k_A \otimes \mathrm{id}^{\otimes j_2-1}_A \otimes \mathrm{id}_B \circ \pi^{k-1|1} \\
&\quad + \sum_{j_1+j+j_2|j_2=0} m^{j_1+j_2|1}_f \circ (\mathrm{id}^{j_1}_A \otimes (m^{j-1|1}_f \circ \pi^{j-1|1} + f^j)) \circ \pi^{j|0})) \\
&= \begin{pmatrix} \sum_{j_1+j+j_2} f^{j_1+1+j_2} \circ (\mathrm{id}^{\otimes j_1}_A \otimes m^j_A \otimes \mathrm{id}^{\otimes j_2}_A) \circ \pi^{k|0} \\ + \sum_{j_1+j+j_2|j_2=0} m^{j_1-1|1}_f \circ (\mathrm{id}^{j_1} \otimes f^j) \circ \pi^{j|0})) \end{pmatrix} \\
&\quad + \begin{pmatrix} \sum_{j_1+j+j_2|j_2\neq 0} m^{j_1+j_2|1} \circ (\mathrm{id}^{\otimes j^1}_A \otimes m^j_A \otimes \mathrm{id}^{\otimes j_2-1}_A \otimes \mathrm{id}_B) \circ \pi^{k-1|1} \\ + \sum_{j_1+j+j_2|j_2=0} m^{j_1+j_2|1}_f \circ (\mathrm{id}^{j_1} \otimes (m^{j-1|1} \circ \pi^{j-1|1})) \circ \pi^{j|0})) \end{pmatrix}
\end{aligned}$$

The first sum gives the quadratic A_∞ homomorphism relations for f, and is therefore zero. The second term is the quadratic A_∞ module relations, is therefore zero. □

A limitation of this cone construction is that it is only defined when the algebras A and B are uncurved. This is due to our inability to construct a change of base homomorphism for curved left A_∞ modules. This limitation can be remedied by studying instead bimodules, and constructing fiber products instead of mapping cones.

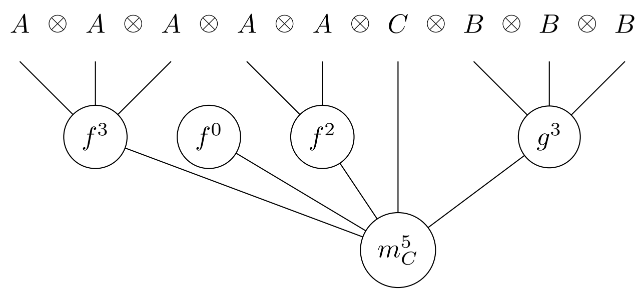


Figure 7: A typical term contributing to $m_{fg}^{5|1|3}$.

3.2 Fiber Product

Definition 3.2.1. *Let A, B be A_∞ algebras. An (A, B)- bimodule is a filtered graded Λ-module M, along with a set of maps*

$$m_{A|M|B}^{k_1|1|k_2} : A^{\otimes k_1} \otimes M \otimes B^{\otimes k_2}$$

satisfying filtered quadratic A_∞ module relations for each pair k_1, k_2

$$\begin{aligned} 0 = & \sum_{\substack{j_1+j+j_2=k_1+1+k_2 \\ j_1+j<k_1}} m_{A|M|B}^{j_1|1|j_2} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m_A^j \otimes \mathrm{id}^{\otimes k_1-j_1-j} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2}) \\ & + \sum_{\substack{j_1+j+j_2=k_1+1+k_2 \\ j_1\leq k_1 \leq j_1+j}} m_{A|M|B}^{j_1|1|j_2} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m_{A|M|B}^{k_1-j_1|1|k_2-j_2} \otimes \mathrm{id}_B^{\otimes j_2}) \\ & + \sum_{\substack{j_1+j+j_2=k_1+1+k_2 \\ k_1<j_1}} m_{A|M|B}^{j_1|1|j_2} \circ (\mathrm{id}_A^{\otimes k_1} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2-j_2-j} \otimes m_B^j \otimes \mathrm{id}_B^{\otimes j_2}) \end{aligned}$$

Again this appears to be three separate sums, but can be restated as one sum in the language of trees.

Remark 3.2.2. *When A and B are uncurved, then a (A, B) bimodule can be made into a left A module by restricting*

$$m_{A|M}^{k|1} := m_{A|M|B}^{k|1|0}.$$

The A_∞ relations follow from the quadratic A_∞ relations for the bimodule where the B-inputs have been evaluated at 0.

It is important to note that this does not *hold if the modules A and B are curved, as m_B^0 terms may contribute to the quadratic A_∞ module relations causing M to* fail to be a left A module!

Claim 3.2.3. *Let C be an A_∞ algebra and let $f : A \to C$ and $g : B \to C$ be filtered A_∞ morphisms. Then C has the structure of a (A, B) bimodule.*

Proof. This only requires that f and g be filtered A_∞ maps of filtered A_∞ algebras. The bimodule structure is given the higher product maps

$$m_{fg}^{k_1|1|k_2} = \sum_{\substack{h_1+\cdots h_{\alpha_1}=k_1 \\ i_1+\cdots+i_{\alpha_2}=k_2}} m_C^{\alpha_1+1+\alpha_2} \circ (f^{h_1} \otimes \cdots \otimes f^{h_{\alpha_1}} \otimes \mathrm{id}_C \otimes g^{i_1} \otimes \cdots \otimes g^{i_{\alpha_2}})$$

These correspond to trees labelled in fig. 7. We show that these satisfy the A_∞ bimodule

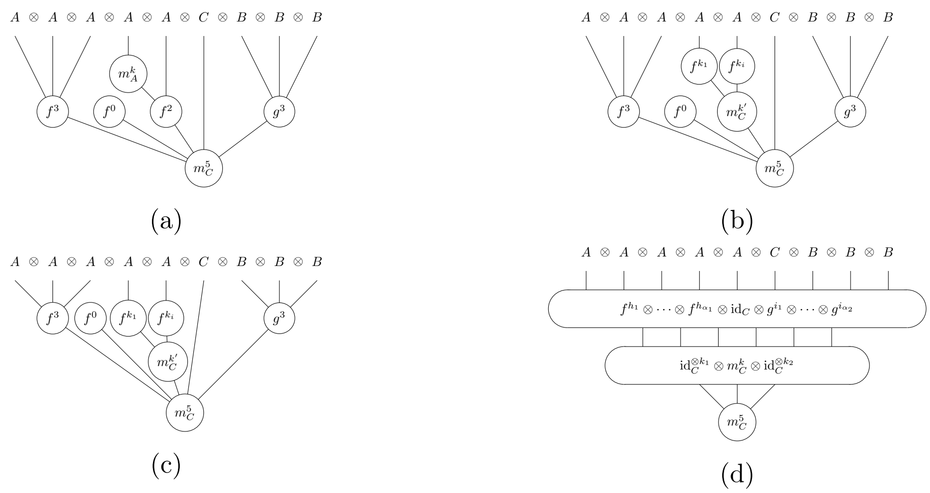


Figure 8: dg the A_∞ change of base bimodule relations.

relations by explicit computation. Examining the terms of the A_∞ relations gives us the following preliminary relations:

$$\sum_{\substack{j_1+j+j_2=k_1+1+k_2\\ j_1+j<k_1}} m^{j_1|1|j_2}_{A|M|B} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m_A^{j} \otimes \mathrm{id}^{\otimes k_1-j_1-j} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2})$$
$$= \sum_{\substack{j_1'+j'+j_2'=\alpha_1+1+\alpha_2\\ \alpha_1<j_1+j'}} m_C^{\alpha_1+\alpha_2+1}(\mathrm{id}^{\otimes j_1'} \otimes m_C^{j'} \otimes \mathrm{id}^{\otimes j_2'}) \circ (f^{h_1} \otimes \cdots \otimes f^{h_{\alpha_1}} \otimes \mathrm{id}_C \otimes g^{i_1} \otimes \cdots \otimes g^{i_{\alpha_2}}).$$

$$\sum_{\substack{j_1+j+j_2=k_1+1+k_2\\ j_1\le k_1\le j_1+j}} m^{j_1|1|j_2}_{A|M|B} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m^{k_1-j_1|1|k_2-j_2}_{A|M|B} \otimes \mathrm{id}_B^{\otimes j_2})$$
$$= \sum_{\substack{j_1'+j'+j_2'=\alpha_1+1+\alpha_2\\ j_1'\le\alpha_1\le j_1'+j'}} m_C^{\alpha_1+\alpha_2+1}(\mathrm{id}^{\otimes j_1'} \otimes m_C^{j'} \otimes \mathrm{id}^{\otimes j_2'}) \circ (f^{h_1} \otimes \cdots \otimes f^{h_{\alpha_1}} \otimes \mathrm{id}_C \otimes g^{i_1} \otimes \cdots \otimes g^{i_{\alpha_2}}).$$

$$\sum_{\substack{j_1+j+j_2=k_1+1+k_2\\ k_1<j_1}} m^{j_1|1|j_2}_{A|M|B} \circ (\mathrm{id}_A^{\otimes k_1} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2-j_2-j} \otimes m_B^{j} \otimes \mathrm{id}_B^{\otimes j_2})$$
$$= \sum_{\substack{j_1+j+j_2=\alpha_1+1+\alpha_2\\ j_1'+j'<\alpha_1}} m_C^{\alpha_1+\alpha_2+1}(\mathrm{id}^{\otimes j_1'} \otimes m_C^{j'} \otimes \mathrm{id}^{\otimes j_2'}) \circ (f^{h_1} \otimes \cdots \otimes f^{h_{\alpha_1}} \otimes \mathrm{id}_C \otimes g^{i_1} \otimes \cdots \otimes g^{i_{\alpha_2}}).$$

We give a graphic explaining these three preliminary relations in fig. 8. Making these substitutions into the quadratic relation we want to prove,

$$\sum_{j_1+j+j_2=k_1+1+k_2} \begin{pmatrix} \sum_{j_1+j<k_1} m_{A|M|B}^{j_1|1|j_2} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m_A^j \otimes \mathrm{id}^{\otimes k_1-j_1-j} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2}) \\ +\sum_{j_1\leq k_1 \leq j_1+j} m_{A|M|B}^{j_1|1|j_2} \circ (\mathrm{id}_A^{\otimes j_1} \otimes m_{A|M|B}^{k_1-j_1|1|k_2-j_2} \otimes \mathrm{id}_B^{\otimes j_2}) \\ +\sum_{k_1<j_1} m_{A|M|B}^{j_1|1|j_2} \circ (\mathrm{id}_A^{\otimes k_1} \otimes \mathrm{id}_M \otimes \mathrm{id}_B^{k_2-j_2-j} \otimes m_B^j \otimes \mathrm{id}_B^{\otimes j_2}) \end{pmatrix}$$

$$= m_B(\mathrm{id} \otimes m_B \otimes \mathrm{id}) \circ (f^{j_1} \otimes \cdots \otimes f^{j_2} \otimes \mathrm{id}_C \otimes g^{i_1} \otimes \cdots \otimes g^{i_{j_2}})$$
$$= 0$$

□

This bimodule construction allows us to construct fiber products in the category of A_∞ algebras.

Claim 3.2.4. *Suppose we have a diagram of A_∞ algebras,*

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

Then $A \cup_C B := A \oplus C[1] \oplus B$ can be given the structure of an A_∞ algebra, called the homotopy fiber product *which fits into the diagram*

$$\begin{array}{ccc} A \cup_C B & \xrightarrow{\pi_A} & A \\ \downarrow \pi_B & & \downarrow f \\ B & \xrightarrow{g} & C \end{array} .$$

Proof. The A_∞ structure on this algebra is similar to that considered for the mapping cone. We denote by

$$\pi_{A|C|B}^{k_1|k|k_2} : (A \oplus C[1] \oplus B)^{\otimes(k_1+k+k_2)} \to A^{\otimes k_1} \otimes (C[1])^{\otimes k} \otimes B^{\otimes k_2}$$

the standard projection. The A_∞ product on $A \oplus C[1] \oplus B$ is given by

$$m_{A\cup_C B}^k = \left((m_A^k \circ \pi_{A|C|B}^{k|0|0}) \oplus \left(f^k \circ \pi_{A|C|B}^{k|0|0} + \left(\sum_{k_1+1+k_2=k} m^{k_1|1|k_2} \circ \pi^{k_1|1|k_2} \right) + g^k \circ \pi^{0|0|k} \right) \oplus m_B^k \circ \pi^{0|0|k} \right)$$

The check that this satisfies the A_∞ relations is similar to definition 3.1.3. □

Remark 3.2.5. *In the category of differential graded algebras, there is a well defined fiber product given by*

$$A \cup_C B := \{(a, b) \mid f(a) = g(b)\}.$$

This definition does not carry over to A_∞ algebras, as this construction implicitly uses the fact that morphisms of DGAs have well defined images. However, a morphism of A_∞ algebras do not have a well defined image, as the homotopies described by the f^k need not lie in the image of f^1.

4 Mapping Cylinders

In the category of chain complexes, there is a dictionary between morphisms and mapping cylinders. In this section, we extend this dictionary to filtered A_∞ algebras.

4.1 Morphisms are Mapping Cylinders

Definition 4.1.1. *Let $f : A^- \to A^+$ be a morphism of A_∞ algebras. Let* id : $A^+ \to A^+$ *be the identity. The* mapping cylinder of f *is the A_∞ fiber product*

$$B_f := A^- \cup_{A^+} A^+.$$

We will denote a mapping cylinder as

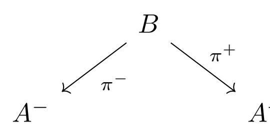
$$A^- \leftrightarrow B_f \to A^+$$

Definition 4.1.2. *Let A^+ and A^- be two filtered A_∞ algebras. A* cylinder *from A^+ to A^- is a filtered A_∞ algebra B which as a vector space is isomorphic to $A^- \oplus A^+[1] \oplus A^+$, and satisfies the following properties.*

- *The chain differential on B is the chain complex mapping cylinder:*

$$\begin{pmatrix} m^1_{A^-} & 0 & 0 \\ f^1 & m^1_{A^+} & \mathrm{id}_{A^+}[1] \\ 0 & 0 & m^1_{A^+} \end{pmatrix}.$$

- *The projections of chain complexes*



can be extended to A_∞ homomorphisms $\pi^k_\pm$, with $\pi^k_\pm = 0$ for all $k \neq 1$.

We denote such a mapping cylinder

$$A^- \leftrightarrow B \to A^+.$$

The cylinders from A^- to A^+ are in correspondence with morphisms $f : A^- \to A^+$.

Theorem 4.1.3 (Cylinders are Mapping Cylinders)**.** *Let A^- and A^+ be two filtered A_∞ algebras.*

1. *To every cylinder $A^- \leftrightarrow B \to A^+$, we can associate a morphism $\Theta_B : A^- \to A^+$.*
2. *To every morphism $f : A^- \to A^+$, we can associate a cylinder*

$$A^- \leftrightarrow B_f \to A^+.$$

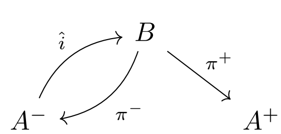
3. *These constructions are compatible in the sense that $\Theta_{B_f} = f$.*

Proof. Each statement is proven using statements from section 2 and section 3.

Proof of item 1 By definition, a mapping cylinder is chain homotopic to its negative end. There exists a chain map

$$i : A^- \to B a \mapsto (a, 0, -f^1(b))$$

The homological perturbation lemma allows us to construct the following associated A_∞ homomorphisms



By taking the composition $\pi^+ \circ \hat{i}^-$, we get a new map from $A^- \to A^+$ called the *pullback-pushforward map,* which we will denote

$$\Theta_B = \pi^+ \circ i^-.$$

Proof of item 2 From construction, the chain structure on B_f fits the definition of a mapping cylinder.

Proof of item 3 It remains to show that $\Theta_{B_f} = f$. An explicit computation suffices. One checks that

$$\pi^+ \circ i^{(T,L)} = \begin{cases} 0 & T \text{ has more than 1 vertex} \\ h \circ m_B^k|_{(A^-)^{\otimes k}} & T \text{ has exactly 1 vertex} \end{cases}$$

and

$$h \circ m_B^k|_{(A^-)^{\otimes k}} = f^k.$$

which shows that the pullback-pushforward map agrees with f. □

4.2 Useful Comments about A_∞ mapping cylinders.

Remark 4.2.1. *There is a small piece of confusing notation here. The mapping cylinder B is said to go from A^+ to A^-. However, the codomain and domain of $f_B : A^- \to A^+$ does not seem to match with this convention. Recall that if X^+ and X^- are two different topological spaces, that a continuous map $\theta : X^+ \to X^-$ has topology $X^- \cup_\theta X^+ \times I$, and gives a map on cohomology from $C^\bullet(A^-) \to C^\bullet(A^+)$. Since we've indexed our A_∞ algebras to be cohomological objects, the induced map from the mapping cylinder goes the opposite direction as expected.*

Filtered A_∞ algebras frequently show up as deformations of honest A_∞ algebras, where we may not have an explicit description of the terms at higher valuations. In many examples, we only want to compute a portion of the A_∞ structure. For this reason, the following corollary is useful.

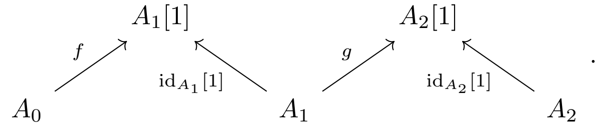
Claim 4.2.2. *Suppose that B satisfies all of the conditions for a mapping cylinder, except we replace*

$$\begin{pmatrix} m^1_{A^-} & 0 & 0 \\ f^1 & m^1_{A^+} & h^{-1} \\ 0 & 0 & m^1_{A^+} \end{pmatrix},$$

where $h^{-1} : B \to B$ is an invertible chain isomorphism. Then there still exists an inclusion $\hat{i} : A^- \to B$, and a pullback-pushforward map $\Theta_B : A^- \to A^+$.

In some situations inspired from geometry, we will for instance know that $h^{-1}_{=0}$, the graded-energy portion of the map h, matches $\mathrm{id}_{A^+}[1]$. This is sufficient to prove that h^{-1} is an isomorphism.

Proposition 4.2.3 (Composition Rule)**.** *Let $f : A_0 \to A_1$ and $g : A_1 \to A_2$ be two A_∞ homomorphism. The* composition cylinder *is defined by gluing two mapping cylinders together:*



The composition cylinder is homotopic to $B_{g\circ f}$.

In many situations, we will know that $h^-1_{=0} = \mathrm{id}_{A^+}[1]$

Proof. We use the homological perturbation lemma to construct a A_∞ homotopy equivalence between

$$\left(\begin{array}{ccccc} A_1[1] & & & & A_2[1] \\ & \nwarrow^{\mathrm{id}_{A_1}[1]} & & \nearrow^{g} & \\ & & A_1 & & \end{array}\right) \sim \left(\begin{array}{c} A_2[1] \\ \\ \\ \end{array}\right)$$

Applying the replacement tool (lemma 2.2.1) produces the cylinder $B_{g\circ f}$ from the composition cylinder, and a homotopy equivalence between these two A_∞ algebras. ☐

Remark 4.2.4. *Given B a mapping cylinder, it is not necessarily the case that $B_{\Theta_B} = B$, as there is more than one way to construct the fiber product structure. We expect that there is a notion of "homotopic relative ends" making B_{Θ_B} equivalent to B. We explore this discrepancy in section 4.3.*

4.3 Example: $A \otimes I$

We look now specifically at the construction of the mapping cylinder of the identity. While this construction can be completely handled using the fiber product that we described before, it is useful from an expository perspective to consider this specific example, which sheds light on how exactly the module structure comes into play in the constructions of A_∞ algebras.

Before describing the A_∞ algebra, we will fix an analogy to differential geometry. Let us suppose that A is the Fukaya-Morse algebra $CM^\bullet(X)$, where X is a smooth compact manifold. We now look at the geometric mapping cylinder of the identity,$X\times[0,1]$. Our construction of the mapping cylinder $A \leftrightarrow B_{\mathrm{id}} \to A$ should describe the Morse cochain complex on $CM^\bullet(X\times[0,1])$. This requires understanding the chain complex $CM^\bullet([0,1])$, and a Künneth formula for the Morse cochain complex.

We use the following model for the A_∞ algebra of the interval.

Definition 4.3.1. *The* interval algebra *is the differential graded algebra generated by*

$$I := \Lambda\langle e^-, x, e^+\rangle$$

where

$$\begin{aligned} \deg(e^-) =& \deg(e^+) = 1 \\ \deg(x) =& 1 \end{aligned}$$

with differential and product structure defined on the basis:

$$\begin{aligned} m^1(e^\pm) =& x \\ m^2(e^\pm, e^\pm) =& e^\pm \\ m^2(x, e^\pm) = -m^2(e^\pm, x) =& x. \end{aligned}$$

There is a well defined tensor product of chain complexes, and so

$$CF^\bullet(X\times[0,1]) = A \otimes I.$$

There is not an immediate way to construct an A_∞ algebra structure on $A\times I$. This in contrast to the setting of differential graded algebras, where there is a canonical tensor product of differential graded algebras. We provide two remarks clarifying the choices made in the construction of a tensor product.

Considerations from Bar Complex A standard trick for A_∞ algebras is to replace them with their bar-complexes,

$$\bar{T}A := A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \cdots,$$

which can be made homotopic to A and are equipped with a dg-coalgebra structure. A_∞ homomorphisms become morphisms of these dg coalgebras.

Many of the constructions that we want to perform on the chain level between A_∞ algebras do not correspond to their chain level counterparts on the bar complex. Even in the settings of DGAs, it is not the case that

$$\bar{T}(A \otimes B) \neq \bar{T}A \otimes \bar{T}B.$$

If A and B are both differential graded algebras, then there is a homotopy equivalence between these two differential graded algebras, although this homotopy equivalence is not canonical. This shows the difficulty of using the bar-construction to define a tensor product.

In the setting of tautologically unobstructed A_∞ algebras, this non-canonical choice can be phrased in terms of picking a simplicial decomposition of the Stasheff associahedra [Lod11]. To our knowledge, this construction has not been extended to curved A_∞ algebras. We expect that these choices are being made in the background of the construction of cones of tautologically unobstructed A_∞ morphisms, and the fiber product of A_∞ algebras.

Considerations about Perturbations. We now give a geometric description for the choices made in constructing the A_∞ mapping cylinder. When defining the Fukaya-Morse algebra $CM^\bullet(X \times [0,1])$, one needs to build a set of perturbations to achieve transversality of the moduli space of trees. The choices of perturbation data are not determined by choices of perturbation data used to define $CM^\bullet(X)$ and $CM^\bullet([0,1])$. As a result, one should not expect there to be some canonical comparison between $CM^\bullet(X \times I)$ and $CM^\bullet(X) \otimes CM^\bullet(I)$.

One (natural, but by no means canonical) choice of perturbation data is to perturb the Morse flow trees in $CM^\bullet(X \times [0,1])$ in "left-to-right" order. This is the perturbation where the amount of perturbation in the $[0,1]$ coordinate applied to a leaf of a flow tree is respects the ordering of the leaves of the tree.

This choice of perturbation corresponds to the following choice of higher products on $A \otimes I$.

Definition 4.3.2. *We say that an element* $\bigotimes_{i=1}^k (a_i \otimes o_i) \in (A \otimes I)^{\otimes k}$ *has* ordered interval component *if*

$$\pi^{0|k}\left(\bigotimes_{i=1}^k (a_i \otimes o_i)\right) = e^- \otimes \cdots \otimes e^- \otimes x \otimes e^+ \otimes \cdots e^+.$$

Claim 4.3.3. *Let A be a curved A_∞ algebra. Define an A_∞ structure on $A \otimes I$ in the following way:*

$$m^k\left(\bigotimes_{i=1}^k a_i \otimes o_i\right) := \begin{cases} m^k(a_1 \otimes \cdots a_k) \otimes e^- & \text{if } o_i = e^- \\ m^k(a_1 \otimes \cdots a_k) \otimes e^+ & \text{if } o_i = e^+ \\ m^k(a_1 \otimes \cdots a_k) \otimes x & \text{if } o_i \text{ is an ordered interval sequence} \\ 0 & \text{otherwise} \end{cases}$$

and the curvature term m^0 by

$$m^0 = m_A^0 \otimes (e^+ + e^-).$$

This is an A_∞ algebra.

Proof. The proof is a computation of the A_∞ relations by hand. We compute

$$\sum_{j_1+j+j_2=k} m^{j_1+1+j_2}(\mathrm{id}^{\otimes j_1} \otimes m^j \otimes \mathrm{id}^{\otimes}\, j_2)\left(\bigotimes_{i=1}^{k} a_i \otimes o_i\right)$$

based on cases of the sequence of $\{o_i\}$.

- Suppose that all of the $o_i = e^+$. Then the A_∞ relation follows trivially from the A_∞ relations on A.
- Similarly, the A_∞ relations hold if all of the o_i are e^-.
- Suppose that the string o_i is ordered. Then for each $j_1 + j + j_2 = k$, the element

$$\left(\bigotimes_{i=1}^{j_1} a_i \otimes o_i\right) \otimes m^j \left(\bigotimes_{i=j_1+1}^{j_1+j} a_i \otimes o_i\right) \otimes \left(\bigotimes_{i=j_1+j+1}^{k} a_i \otimes o_i\right)$$

  again has ordered interval component. Therefore, this reduces to the A_∞ relation on A.
- The string o_i is not interval ordered, all e^+ or all e^0. Then the contracted string

$$\left(\bigotimes_{i=1}^{j_1} a_i \otimes o_i\right) \otimes m^j \left(\bigotimes_{i=j_1+1}^{j_1+j} a_i \otimes o_i\right) \otimes \left(\bigotimes_{i=j_1+j+1}^{k} a_i \otimes o_i\right)$$

  is not interval ordered, all e^+ or all e^-. The product evaluated on this term must be zero.

□

5 Homotopies of Chain Maps

In this section, we show how to recover the definition of a homotopy of A_∞ homomorphisms from our mapping cylinder constructions.

Definition 5.0.1. *Let $f_-, f_+ : (A, m_A^k) \to (B, m_B^k)$ be a pair of A_∞ homomorphisms. A A_∞ homotopy between f_- and f_+ is a set of maps $h^k : A^{\otimes k} \to B[-1]$ satisfying the curved A_∞ homotopy relations*

$$\sum_{j_1+i+j_2=k} \pm h^{j_1+j_2+1}(\mathrm{id}^{\otimes j_1} \otimes m_A^i \otimes \mathrm{id}^{\otimes j_2}) = f_-^{j_1} - f_+^{j_2} + \sum_{\substack{i_1^1+\cdots+i_1^m=j_1\\ i_2^1+\cdots+i_2^n=j_2}} \pm m_B^j(f_-^{i_1^1} \otimes \cdots \otimes f_-^{i_1^m} \otimes h^i \otimes f_+^{i_2^1} \otimes \cdots f_+^{i_2^n})$$

A chain homotopy is filtered *if h is a filtered map. A homotopy is* weakly filtered *if $h^1 m_A^0$ has a positive valuation. The energy loss of a homotopy is least upper bound c so that h^k increases the filtration by at most $k \cdot c$.*

Given an A_∞ algebra B, there are canonical projections $\pi_\pm : B \times I \to B$.

Proposition 5.0.2. *Let $f_-, f_+ : A \to B$ be two A_∞ homomorphisms. There exists a homotopy between f_- and f_+ if and only if there exists an A_∞ homomorphism $f_\pm : A \to B \otimes I$ so that $\pi_- \circ f_\pm = f_-$ and $\pi_+ \circ f_\pm = f_+$.*

Proof. For this example, we use notation from the explicit construction of the A_∞ structure on $B \otimes I$ from claim 4.3.3. We first show that an A_∞ homomorphism $f_\pm : A \to B \otimes I$ gives an A_∞ homotopy between $\pi_- \circ f_\pm$ and $\pi_+ \circ f_\pm$. Define the map

$$h^k := \pi_{B \otimes x} \circ f^k : A^{\otimes k} \to B[1].$$

We also look at the preliminary equalities Starting with the preliminary equalities

$$\begin{aligned}
\pi_{B\otimes e^+} \circ f_\pm =& f_+ \\
\pi_{B\otimes e^-} \circ f_\pm =& f_- \\
\pi_B \otimes x m^j_{B\otimes E} =& m^j_B(\pi_{B\otimes e^-} \otimes \cdots \otimes \pi_{B\otimes e^-} \otimes \pi_{B\otimes x} \otimes \pi_{B\otimes e^+} \otimes \cdots \otimes \pi_{B\otimes e^+}) \otimes x
\end{aligned}$$

We look at $B \otimes x$ component of the quadratic A_∞ homomorphism relations.

$$\pi_{B\otimes x} \sum_{j_1+j+j_2=k} f_\pm^{j_1+j+j^2}(\mathrm{id}_A^{\otimes j_1} \otimes m_A^j \otimes \mathrm{id}_A^{\otimes j_2}) = \pi_{B\otimes x} \sum_{i_1+\cdots+i_j} m^j_{B\otimes I}(f^{i_1} \otimes \cdots f^{i_j}).$$

The left hand side of this relation is the left hand side of the A_∞ homotopy relations:

$$\pi_{B\otimes x} \sum_{j_1+j+j_2=k} f_\pm^{j_1+j+j^2}(\mathrm{id}_A^{\otimes j_1} \otimes m_A^j \otimes \mathrm{id}_A^{\otimes j_2}) = \sum_{j_1+j+j_2=k} h^{j_1+j+j_2}(\mathrm{id}_A^{\otimes j_1} \otimes m_A^j \otimes \mathrm{id})A^{\otimes j_2}).$$

Similarly, the right hand side of the A_∞ homomorphism relation gives the right hand side of the A_∞ homotopy relations.

$$\begin{aligned}
&\pi_{B\otimes x} \sum_{i_1+\cdots+i_j} m^j_{B\otimes I}(f^{i_1} \otimes \cdots f^{i_j}) \\
&= f^- + f^+ + \sum_{i_1^-+\cdots i_{j_1}^-+j+i_1^+ +\cdots i_{j_2}^+=k} m_B^{j_1+1+j_2}(f_-^{i_1^-} \otimes \cdots \otimes f_-^{i_{j_1}^-} \otimes h^j \otimes f_+^{i_1^+} \otimes \cdots f_2^{i_{j_2}^+})
\end{aligned}$$

The left hand side and right hand side together give us the A_∞ homotopy relations. A similar argument shows the reverse direction. □

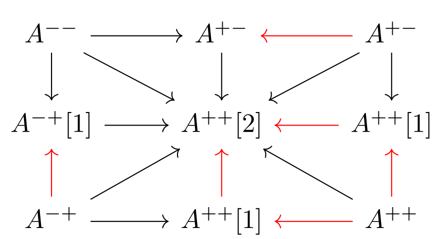
With this viewpoint, the A_∞ homotopy equivalence can be described by maps to cylinders $B \times I$. Note that by section 4.3, there is not a canonical choice of A_∞ structure on $B \otimes I$, so the A_∞ homotopy relations constructed implicitly rely on the choice of A_∞ structure chosen for the cylinder. One take away from this discussion is that the A_∞ homotopy relations are not canonical, but exist up to some kind of homotopy.

The following A_∞ algebra shows up frequently in nature and can be a useful way to build homotopies between A_∞ homomorphisms.

Definition 5.0.3. *Let A^{--}, A^{-+}, A^{+-}, and A^{++} be four filtered A_∞ algebras. Let*

$$\begin{aligned}
A^{--} &\leftrightarrow B^{-\pm} \to A^{-+} \\
A^{--} &\leftrightarrow B^{\pm-} \to A^{+-} \\
A^{+-} &\leftrightarrow B^{+\pm} \to A^{++} \\
A^{-+} &\leftrightarrow B^{\pm+} \to A^{++}
\end{aligned}$$

be four mapping cylinders. A homotopy square $B^{\pm\pm}$ with edges $B^{-\pm}, B^{\pm-}, B^{+\pm}, B^{\pm+}$ is a filtered A_∞ algebra, which as a vector space decomposes as



and presents as a mapping cylinder in two ways:

$$B^{-\pm} \leftrightarrow B^{\pm\pm} \to B^{+\pm}$$
$$B^{\pm-} \leftrightarrow B^{\pm\pm} \to B^{\pm+}$$

where the maps are the obvious projections.

Lemma 5.0.4 (Square Lemma)**.** *Let $B^{\pm\pm}$ be a homotopy square with edges $B^{-\pm}, B^{\pm-}, B^{+\pm}, B^{\pm+}$. Let*

$$\Theta^{-\pm} : A^{--} \to A^{-+}$$
$$\Theta^{\pm-} : A^{--} \to A^{+-}$$
$$\Theta^{+\pm} : A^{+-} \to A^{++}$$
$$\Theta^{\pm+} : A^{-+} \to A^{++}$$

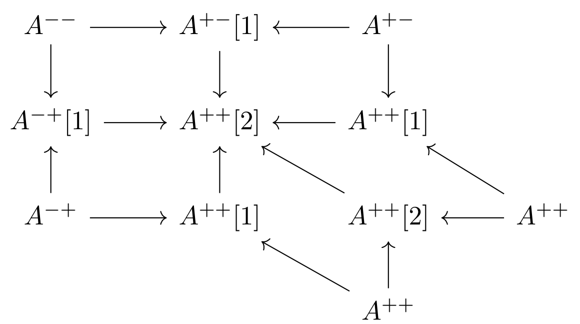
be the four pullback-pushforward A_∞ homomorphisms associated to the four edge mapping cylinders. There is a A_∞ homotopy between

$$\Theta^{\pm+} \circ \Theta^{-\pm} \sim \Theta^{+\pm} \circ \Theta^{\pm-}.$$

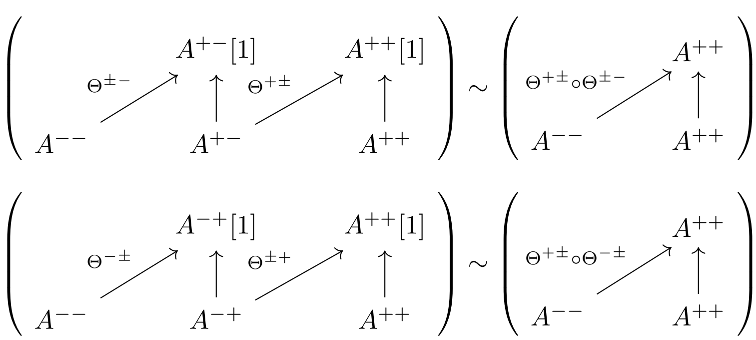
Proof of A_∞ square lemma. By using the replacement tool, 2.2.1 we can replace A^{++} with the homotopic cylinder

$$A^{+} + \times I \sim A^{++} \leftrightarrow A^{++}[1] \leftarrow A^{++}$$

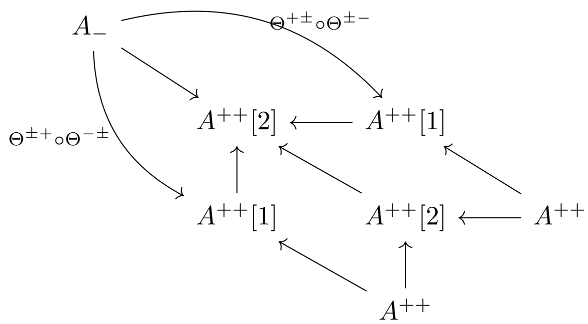
giving us the homotopic complex



We can construct the following homotopy equivalences using the composition rule for mapping cylinders proposition 4.2.3,



Applying the replacement tool, we can replace the sides of the homotopy square:



This gives us the structure of a mapping cylinder for a morphism $\Theta^{\pm\pm} : A^{--} \to A^{++} \otimes I$. By Proposition 5.0.2, $\Theta^{\pm+} \circ \Theta^{-\pm} \sim \Theta^{+\pm} \circ \Theta^{\pm-}$ are A_∞ homotopic. □

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