Tropical Lagrangians in Mirror Symmetry

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Summary

Background: Mirror Symmetry

Background: Complex to Tropical Correspondance

Lagrangian to Tropical Correspondance

- Construction
- Properties of Lagrangian Submanifolds

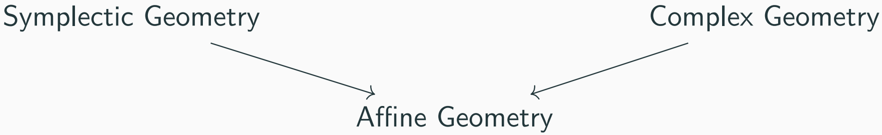
Background: Mirror Symmetry

Mirror Symmetry

Mirror Symmetry is a duality between symplectic geometry on a manifold X and complex geometry on a *mirror manifold* $\check{X}$.

- What is the mechanism which interchanges these two types of manifolds?
- What portions of the geometry are interchanged?

Symplectic $\leftrightarrow$ Affine $\leftrightarrow$ Complex



From Affine to ω and J

Definition

An integral affine manifold Q is a manifold possessing charts whose transitions are all integral affine transitions.

Main feature: Lattices inside of (co)tangent bundle.

$$T^*_{\mathbb{Z}}Q \subset T^*Q$$
$$T_{\mathbb{Z}}Q \subset TQ$$

Building the complex and symplectic manifolds

Remark

T^*Q has a symplectic structure. TQ has an almost complex structure.

$$(X, \omega) := (T^*Q / T^*_{\mathbb{Z}}Q, dp \wedge dq).$$

$$(\check{X}, J) := (TQ / T_{\mathbb{Z}}Q, J).$$

Example (Running Example)

If $Q = \mathbb{R}^n$, then $X = \check{X} = (\mathbb{C}^*)^n$.

SYZ Mirror Symmetry

Conjecture (Strominger, Yau, and Zaslow 1996)

When two spaces X and $\check{X}$ have dual Lagrangian almost torus fibrations over common base Q, their symplectic and complex geometry are interchanged.

$$X \xrightarrow{\text{val}} Q \xleftarrow{\check{\text{val}}} \check{X}$$

Lagrangian submanifolds of $X \leftrightarrow$ Complex Submanifolds $\check{X}$

Context: HMS

More precisely, homological mirror symmetry (Kontsevich 1994) predicts the equivalence of the following categories:

- The Fukaya category of X, whose:
  - Objects are Lagrangian submanifolds and
  - $\hom(L, K)$ is the Lagrangian intersection Floer homology
- Derived Category of Coherent sheaves on $\check{X}$.

Naïve interchanging

Let $V \subset Q$ be a submanifold with "integral tangent space" so that

$$T_{\mathbb{Z}}Q \cap TV$$

is a full lattice. (Think: tangent space has rational slope.)

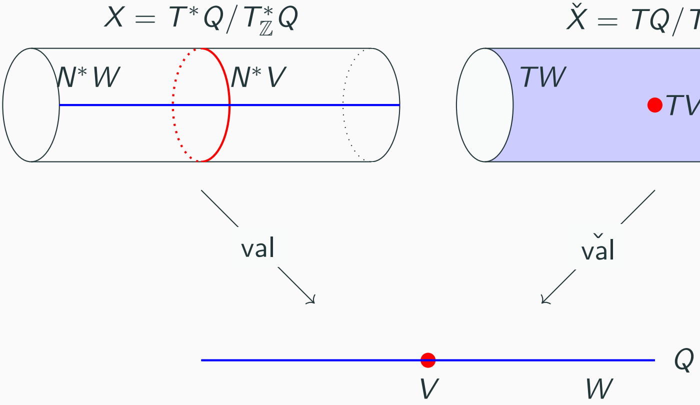
- Conormal subbundle gives us a Lagrangian submanifold

$$N^*V/N^*_{\mathbb{Z}}V \subset X$$

- Tangent subbundle to V gives us an almost complex submanifold

$$TV/T_{\mathbb{Z}}V \subset \check{X}.$$

Symplectic - Tropical - Complex Correspondence on $\mathbb{C}^*$



Background: Complex to Tropical Correspondence

Background: Tropical Geometry

Definition

A tropical polynomial is a piecewise linear concave function

$$\phi : Q = \mathbb{R}^n \to \mathbb{R},$$

with

$$d\phi(q) \in T^*_{\mathbb{Z}} Q$$

whenever $d\phi(q)$ is defined.

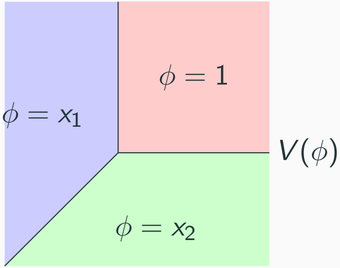
Definition

The tropical hypersurface of ϕ is the subset $V(\phi) \subset Q$ where ϕ is non-linear.

Example: Tropical Pair of Pants

Consider the function

$$\phi(x_1, x_2) = \min(1, x_1, x_2)$$
$$=: 1 \oplus x_1 \oplus x_2.$$



Away from the middle bit, $V(\phi)$ looks like an integral affine submanifold, so we seem to be in good shape to run our previous construction.

Tropicalization

The valuation map relates complex on $(\mathbb{C}^*)^n$ with tropical geometry on $\mathbb{R}^n$.

$$\check{\text{val}}(z_1, \cdots, z_n) := (\log|z_1|, \cdots, \log|z_n|)$$

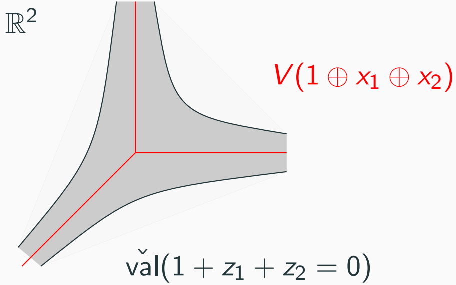
Idea: logarithms allow us approximate polynomials by leading order behaviour

$$\begin{aligned}\log(r_1 + r_2) &\sim \max(\log r_1, \log r_2)\\ \log(r_1 r_2) &= r_1 + r_2\end{aligned}$$

which in turn yields piecewise linear geometry.(*)

Example: Pair of Pants

The pair of pants $1 + z_1 + z_2 = 0 \subset (\mathbb{C}^*)^2$ projects to a *amoeba* of the tropical pair of pants $V(1 \oplus x_1 \oplus x_2) \subset \mathbb{R}^2$.



Lagrangian to Tropical Correspondence

Tropical Lagrangians

Notation

From here on, $X = (\mathbb{C}^*)^n$ and $Q = \mathbb{R}^n$.

Theorem (H)

To every tropical polynomial $\phi : Q \to \mathbb{R}$, we can build a Lagrangian submanifold $L(\phi) \subset X$ with valuation in a neighborhood of the tropical variety,

$$\text{val}(L(\phi)) \subset B_\epsilon(V(\phi)).$$

Furthermore, the Lagrangian $L(\phi)$ is unobstructed, and gives an object in an appropriate (monomial admissible) Fukaya category.

Compare to recent work of Matessi 2018; Mikhalkin 2018.

Idea of Construction

1. Start by smoothing tropical polynomial $\tilde{\phi} : Q \to \mathbb{R}$.
2. Build Lagrangian sections $\sigma_\phi : Q \to X$ by taking the composition

$$Q \xrightarrow{d\tilde{\phi}} T^*Q \xrightarrow{/T^*_{\mathbb{Z}}Q} X.$$

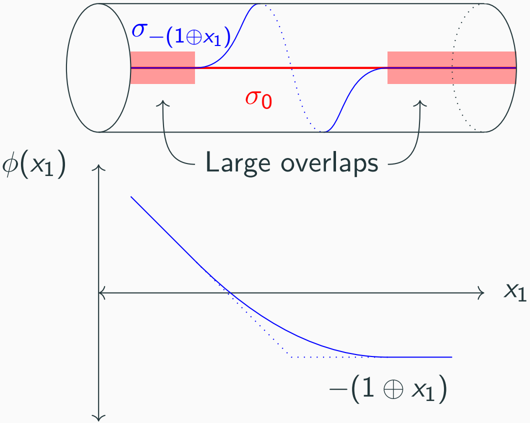
   Originally used in Abouzaid 2009.
3. Observation: Whenever $d\tilde{\phi}(q) \in T^*_{\mathbb{Z}}Q$

$$\sigma_\phi(q) = \sigma_0(q),$$

   So our sections have a large amount of overlap.

Example on $\mathbb{C}^*$

$\mathbb{C}^* = T^*\mathbb{R}/T^*_{\mathbb{Z}}\mathbb{R}$



Build Lagrangian sections

$$Q \xrightarrow{d\tilde{\phi}} T^*Q \xrightarrow{/T^*_{\mathbb{Z}}Q} X.$$

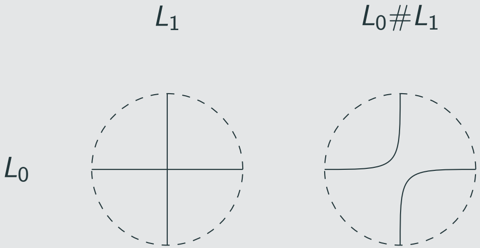
Whenever $d\tilde{\phi}(q) \in T^*_{\mathbb{Z}}Q$

$$\sigma_\phi(q) = \sigma_0(q),$$

Lemma

Under certain conditions (which are favorable here), when Lagrangian submanifolds L_0 and L_1 have intersection U, there exists a Lagrangian submanifold $L_0 \#_U L_1$ which lives in a neighborhood of the symmetric difference of L_0 and L_1.

Example



Construction of Tropical Lagrangian Submanifolds

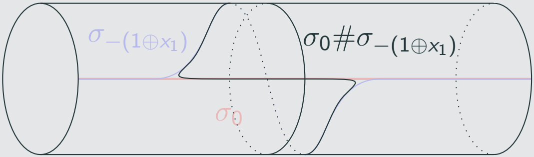
Definition

Given ϕ a tropical polynomial, we define the tropical Lagrangian submanifold

$$L(\phi) := \sigma_0 \#_U \sigma_{-\phi}.$$

Example

$$F_0\mathbb{C}^* \; \sigma_0\#\sigma_{-(1\oplus x_1)} =: L(1\oplus x_1)$$



Some Geometric Properties

- Up to the equivalence relation of Hamiltonian isotopy, our construction of $L(\phi)$ is independent of choices (other than ϕ, of course!)
- We can make the image of $L(\phi)$ under the valuation map

$$\mathrm{val} : X \to Q$$

lie in an arbitrarily small neighborhood of $V(\phi)$.

Floer Theoretic Properties of $L(\phi)$

If we want to use $L(\phi)$ for Lagrangian intersection Floer theory, need to know a little bit more.

- The Lagrangians $L(\phi)$ are rarely monotone.
- There are choices of complex structure for which $L(\phi)$ bounds regular Maslov index zero disks!
- This is not a fluke: the count of these disk provide an important role to corrections of the Floer theory of these Lagrangians.

Lemma

Next best thing: The Lagrangians $L(\phi)$ are unobstructed (by bounding cochain.)

Sketch of Proof

- Take a sequence of Hamiltonian isotopic L_k, starting with $L(\phi)$, which approaching a subset of $\sigma_0 \cup \sigma_{-\phi}$.
- If holomorphic disks with bounded energy exist for all L_k, we can construct a holomorphic polygon with boundary on $\sigma_0 \cup \sigma_{-\phi}$.
- No such polygons exist on $\sigma_0 \cup \sigma_{-\phi}$ for index reasons.

This is enough to show that the count of disks can be cancelled by bounding cochain.

Necessity of Surgery construction and/or bounding cochains

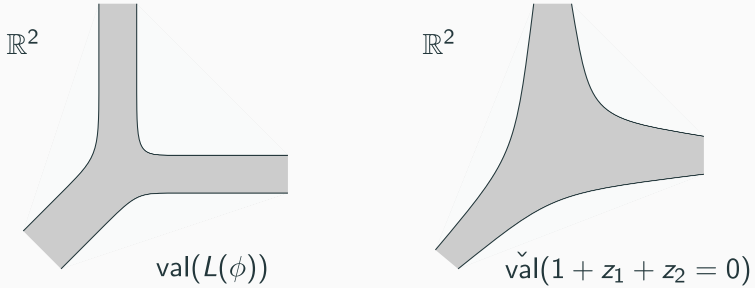
Remark

There exist tropical Lagrangians built from pair of pants decompositions, for which:

- *We do not expect to be able to build from Lagrangian surgery and,*
- *Are obstructed as objects of the Fukaya category.*

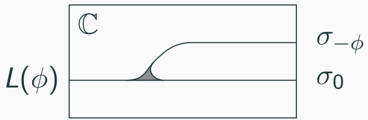
Theorem (H)

Under this matching, Lagrangians $L(\phi)$ can be shown to be mirror to sheaves $\mathcal{O}_D$ with support on a divisor D.



Sketch of Proof

There exists a Lagrangian cobordism with ends $K : (\sigma_0, \sigma_{-\phi}) \rightsquigarrow L(\phi)$.



Biran and Cornea 2014 gives an exact sequence in Fuk(X),

$$\sigma_0 \to \sigma_{-\phi} \to L(\phi)$$

From Abouzaid 2009; Hanlon 2018, σ_0, σ_ϕ are mirror to $\mathcal{O}, \mathcal{O}(-D)$.

$$\mathcal{O} \to \mathcal{O}(-D) \to \mathcal{O}_D.$$

Take away

- SYZ mirror symmetry tells us that we are supposed to obtain mirrors by passing to tropical geometry.
- Homological mirror symmetry tells us that sheaves supported on a complex hypersurface are mirror to Lagrangians in Fukaya category.
- This construction shows that both items are tied together in some concrete geometric fashion.

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