Legendrian Contact Homology in TCY3-folds

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Abstract

Here is a set of notes I wrote to get a better understanding of the mirror surface to a Toric Calabi-Yau 3-fold in terms of the Legendrian contact homology. In Sections 1, 2, and 3, we review Complex, Symplectic and Contact geometries related to Toric Calabi-Yau 3-folds. In Section 4, we introduce Legendrian Contact homology as a tool for understanding Lagrangian fillings. In Section **??**, we use compute the Legendrian contact homology of a torus in the boundary of a TCY 3-fold.

1 Toric Varieties

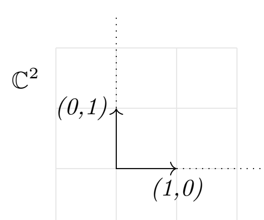
We'll start by taking a complex geometry viewpoint on toric geometry. An excellent source for more information about toric varities is [CLS11].

Definition 1 (Complex Toric Manifold)**.** *A complex toric manifold is a irreducible variety X along with an open subset $(X^0) \subset X$ isomorphic to $(\mathbb{C}^*)^n$ so that the action of $(\mathbb{C}^*)^n$ on itself extends to an action on X.*

We call $(\mathbb{C}^*)^n$ the *big torus*, and its action on X gives us a combinatorial characterization of a toric variety by looking at the limiting values of the action. A toric action tells us that there are many ways to "probe" our space by looking at the image of the orbit of a point under the action of $\mathbb{C}^*$.
We should think of each toric variety as corresponding to the "big torus," plus some lower dimensional strata we need to add in order to close it up. Each of these lower dimensional strata are in the fixed point set of some non-identity element. By recording the combinatorics of the compactification of $(\mathbb{C}^*)^n$ to X, we can recover a lot data about X in a short form.

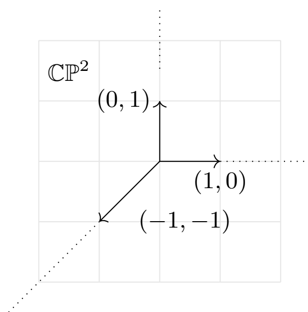
Example 1. *Here are a few examples:*

- *$\mathbb{C}^n$ is toric variety, with the standard action of $(\mathbb{C}^*)^n$. In the case of $\mathbb{C}^1$, notice that there is a single fixed point of the toric action at the origin. In higher dimensional cases, there are strata that are fixed by some element of the toric action.*
  If we look at the 2 dimensional case, we represent the fixed points by the following diagram in the plane:

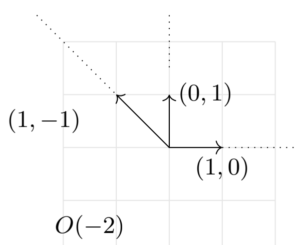


  This map tells us to add in 2 stratum, one for the points fixed under multiplication on the first coordinate, and one for points fixed by multiplication on the second coordinate. Since these arrows point in the positive direction, these are points added near the "origin" of the toric action.
  The union of fixed points over all non-identity elements of $(\mathbb{C}^*)^n$ is given by the polynomial $z_1 z_2 \cdots z_n = 0$.

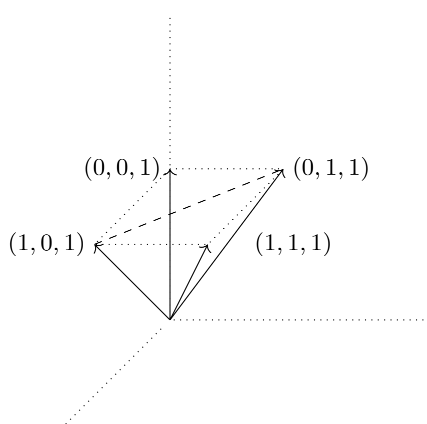
- $\mathbb{CP}^n$ *is a toric variety. In the example of* $\mathbb{CP}^1$*, the toric action corresponds to rotation about the axis, with 2 fixed points, one at each pole. In the higher dimensional case, the homogeneous polynomial* $z_1 \dots z_k = 0$ *also cuts out strata of fixed points.*
  If we again look at the 2 dimensional case, we have several submanifolds that are fixed elements of group. They correspond to points in the 3 hyperplanes. If a point is given by coordinates $(z_1, z_2, 1)$*, the points fixed by the action are given by the multiplication on the first coordinate, multiplication on the second coordinate, and multiplication on the pair of the first two coordinates. We need to add toric divisors as the first coordinate goes to 0, as the second coordinate goes to 0, and as both the first and second coordinate go to* ∞*. The combinatorics of these three strata is captured in the following diagram :*



- *The cotangent bundle of* $\mathbb{CP}^1$ *is a toric variety. This can best be seen as the bundle* $O(-2) \to \mathbb{CP}^1$*, and (thinking of this as a cotangent bundle) the fixed strata are the 0 section and the fibers above the north and south pole. The resulting compactification is given by the diagram*



- *A example that we'll frequently look at is the* resolved conifold, $O(-1) \oplus O(-1) \to \mathbb{CP}^1$. *This can be described the toric fan :*



  which has 2 cones of highest dimension.

The above intuition can be made rigorous using the algebraic machinery of fans.

Definition 2 (Character)**.** *Let* T^n *be an algebraic torus. A* character *of* T^n *is a holomorphic map* $\chi : T^n \to \mathbb{C}^*$ *which is a homomorphism of toric action.*

The characters form a group, with operation given by:

$$(\chi_u + \chi_v)(x) = \chi_u(x) \cdot \chi_v(x).$$

We call this group the *character lattice* and denote it as N.

Definition 3. *A* cocharacter *of the torus is a map $\lambda_u : (\mathbb{C}^*) \to T^n$ which is a homomorphism of the toric action. We denote the cocharacter lattice M.*

The character and cocharacter lattice have a natural pairing which records the degree of the composition.
In a compact toric variety, every map $\mathbb{C}^* \to X$ extends to a map from $\mathbb{C} \to X$. As such, we can partition the cocharacters of X based on their limiting values. This decomposition breaks the cocharacter lattice in to convex components: if λ_u and λ_v have the same limiting value, then λ_{u+v} will also have the same limiting value. We call each strata of this grouping a *cone*, and the arrangement of all of the cones the *fan* of a toric variety, denoted $\Sigma \subset N$.

Theorem 1. *The data of a fan (up to $SL(\mathbb{Z}^n)$) is equivalent to the data of a toric variety.*

Claim 1. *A toric variety is smooth if and only if the generators of each maximal cone form a basis for the space.*

Remark 1. *Notice that an affine toric variety cannot have an affine semigroup which contains an v and $-v$.*

A lot of the data of the toric variety are hidden in the fan.
Recall that a divisor is a linear combination of hypersurfaces. A natural set of hypersurfaces come from extending the characters of the toric action to $\mathbb{C}$. These divisors are exactly given by the compactifications, and correspond to the 1-dimensional cones of the fan of X.

Claim 2. *These toric divisors generate the set of all divisors.*

The *Canonical Bundle* Ω^n is the nth exterior power of the cotangent bundle on X. Since the toric divisors generate all divisors, we can express the canonical divisor in terms of the combinatorics of the fan.

Claim 3. *The canonical divisor is given by the negative sum of all toric divisors.*

This representation gives us an easy to check condition for triviality of the canonical bundle.

Claim 4. *The canonical bundle is trivial if and only if there exists $w \in M$ so that $\langle w, v\rangle = 1$.*

When the canonical bundle of a variety is trivial we call the manifold *Calabi-Yau.* As a result, whenever we are in the Toric Calabi-Yau case, we can make a choice of character w with the following properties:

- The fan Σ can be constructed so that all of the 1 -dimensional cones end on a hyperplane defined by w.
- The combinatorics of the fan can be reduced to a projection onto that fan. WE call this the *toric diagram* of X.
- The toric variety comes with a map $\chi_w : X \to \mathbb{C}$, where the critical locus of this map corresponds to the toric divisors.

2 Symplectic Geometry

The theory of Symplectic Toric geometry and it's interface with complex geometry is presented in a very nice form in [Sil01]. The relation between Calabi-Yau geometry and toric manifolds I learned from the notes of [Clo09]. We introduce the protagonist of our story:

Definition 4. *A symplectic manifold is a $2n$ dimensional manifold X along with a choice of* symplectic form *$\omega \in \Omega^2(X, \mathbb{R})$ so that $d\omega = 0$ and ω^n is everywhere nonzero.*

Symplectic manifolds admit particularly nice reductions under groups which act on them via symplectomorphism.

Definition 5. *Let G be a lie group acting symplectically on X. Then there is a moment map $\mu : G \to \mathfrak{g}^*$, whose coordinates on $\mathfrak{g}^*$ are locally given by hamiltonian flow.*

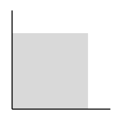
The action of a torus on a symplectic manifold gives a particularly nice result:

Theorem 2 (M. Atiyah and V. Guillemin and S. Sternberg). *If G is a n torus, then the image of the moment map a convex hull of finitely many points.*

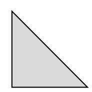
The general fiber of such a moment map is a Lagrangian submanifold.

Example 2. *Here are some examples of moment maps.*

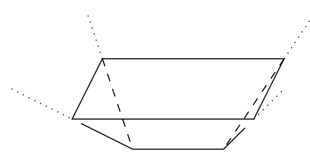
- $\mathbb{C}^n$. *Here, the action is given by multiplication on the coordinates. Again, notice that the action degenerates over each of the strata of $z_1 z_2 \cdots z_n = 0$. In the case of $\mathbb{C}^2$, the moment map of this is a subset of $\mathbb{R}^2$ given by the 1st quadrant.*



- $\mathbb{CP}^n$. *The torus action in this case is again given by multiplication on the homogeneous coordinates. Notice that $\mathbb{CP}^n$ has $n+1$ affine charts, each of which have compatible torus actions. When stitched together, the moment map of $\mathbb{CP}^2$ is given by a triangle:*



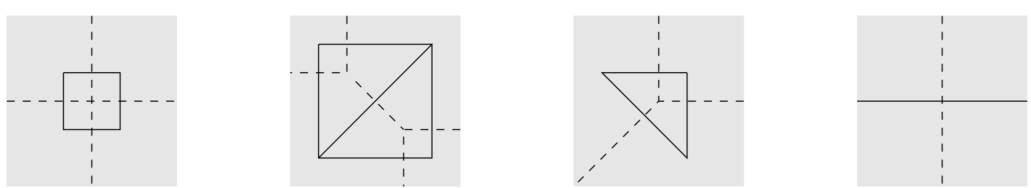
- $O(-1) \oplus O(-1) \to \mathbb{CP}^1$. *This example is a little difficult to draw as it sits in 3 dimensions, but it is given by a "prism" like polytope*



These points tell us how the symplectic toric action degenerates.

Claim 5. *Let X be a $2n$ symplectic manifold with the symplectic action of a T^n on it. Then the dual fan to the momement polytope is the toric fan of X.*

In the case where our manifold is additionally known to be Calabi-Yau, this tells us that there exists a vector which pairs nondegenerately with the normals to each face of the moment polytope. As a result, there is a projection of the boundary of this polytope to a hyperplane of one dimension lower. This is the *pq-web*, and it is dual to the toric diagram.



Conifold | $O(-1) \oplus O(-1)$ | $\mathbb{C}^3$ | $\mathbb{C}^2 \times \mathbb{C}^*$

In the case of a Calabi-Yau 3-fold, this pq-web allows us to read of the structure of the critical locus of χ_w very quickly.

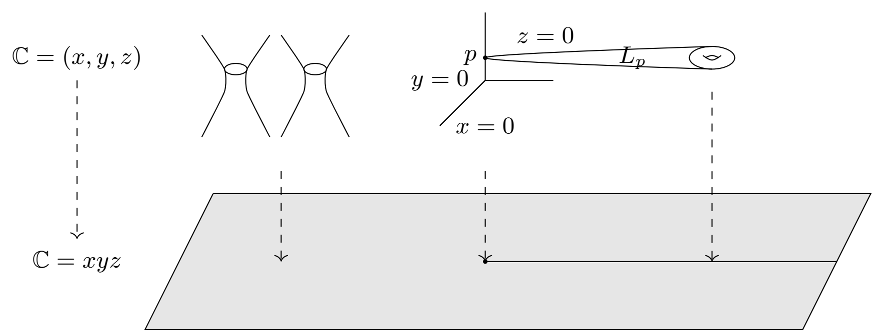
- Each line is a $\mathbb{C}^*$
- Each semi-infinite edge is a copy of $\mathbb{C}$.
- Each edge is a $\mathbb{CP}^1$.

The critical locus is a normal-crossings divisor, and away from the crossings has the structure of a Lefschetz-bott fibration.
The symplectic form ω gives us a connection on the fiber bundle $\chi_w : X \to C$, called the *symplectic connection.* We can use this symplectic connection to construct many different families of Lagrangian submanifolds.

Claim 6. *Let L be a lagrangian submanifold of a fiber $\chi_w^{-1}(p)$. The parallel transport of a Lagrangian L over a path in the base $\mathbb{C}$ is a Lagrangian of the total space X.*

Example 3. *In the toric case, the general fiber is symplectomorphic to $(\mathbb{C}^*)^{n-1}$, so it is natural to look at the product tori in the fiber. The parallel transport of these Lagrangians over circles in base give us new lagrangian tori in the toric manifold. One can use this construction to create SYZ fibrations for toric Calabi-Yaus, which is the geometric construction for constructing mirror manifolds.*

Let $\mathrm{Crit}(\chi_w)^0$ be the portion of the critical locus away from the crossings. This is a Kähler submanifold, and to each lagrangian (in this case, curve) of this submanifold, we can associate a *lagrangian thimble* of X.
Since each edge of the pq web corresponds to the image of the moment map from the critical locus, each point p in pq-web gives us a special Lagrangian $S^1 \subset \mathrm{Crit}(\chi_w)^0$. We call the corresponding special lagrangian thimble the *Aganagic-Vafa brane*, and denote it L_p. These Lagrangians were first described in [HL82], and are therefore sometimes called *Harvey-Lawson branes.*



Conjecture 1 (Mirror Symmetry)**.** *The corrected moduli space of the AV branes should be given a hypersurface Σ_X, which is a thickening of the pq-web.*

While the conjecture was initially motivated by physical intuition, there are 3 different mathematical sources of evidence: microlocal sheaves [TZ16] , open Gromov-Witten invariants [LLLZ09], and Legendrian Contact homology [AV00], [Ekh13]. Today, we'll be looking at the third route.

2.1 Disks in Product Tori

We now take a brief look at the disks in the Lagrangian fiber of the moment polytope.

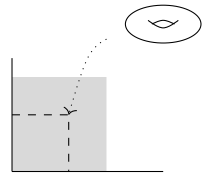
Claim 7. *Let F_p be a fiber of the moment map of a toric symplectic manifold. Eor every toric divisor D, and every point $z \in F_p$ there exists exactly one maslov index 2 disk with boundary in F_p with boundary passing through z and intersecting transversely with D. Furthermore, these are the only Maslov index 2 disks.*

We can visualize these disks by looking at their images under the moment map, which look like "tropical-amoebas."

Example 4. *The standard product torus in $\mathbb{C}^2$ is given by $(e^{i\theta_1}, e^{i\theta_2})$. This is the fiber at $(1,1)$ of the moment map . The only disks that this bounds are*

$$z \mapsto (a, e^{i\theta_2})$$
$$z \mapsto (e^{i\theta_1}, z)$$

In the moment polytope, we see these three disks as "lines" extending from the product torus perpendicularly to each of the boundary strata– which correspond to the toric divisors.



It is frequently convenient to encode this data in the form of a generating function.

Definition 6. *Let $R = \mathbb{C}[[H_2(X, F_p]]$. Define the function $F_p(T_i) := \sum_{\beta \in H_2(X,F_p)} T^\beta$.*

In cases with good convergence properties, we can make this a generating function over the complex numbers by

$$F_p^\omega(T_i) = \sum_{\beta \in H_2(X,F_p)} e^{-\omega(\beta)}.$$

In this case, it is also useful to note that elements of $H_2(X, F_p)$ can be expressed in terms of $H_1(F_p)$ and $H_2(X)$. In the case where $H_2(X) = 0$, we have that every homology class is some homology class coming from the torus. Notationally, let $y_1, \ldots, y_n$ be the generators of $H_1(F_p)$, and let $q_1, \ldots, q_k$ the generators of $H_2(X)$.

Example 5. *In the case of $\mathbb{C}^3$, this generating function is given by*

$$F(y_1, y_2, y_3) = T^{y_1} + T^{y_2} + T^{y_3}$$

Rescaling this by a factor of $u^{-1} = T^{y_3}$ gives

$$uF = T^{y_1+y_3} + T^{y_2+y^3} + 1$$

Example 6. *In the case of the resolved conifold, let q be the class of $\mathbb{CP}^1$. We have 4 toric divisors, giving us the generating function*

$$T^{y_1} + T^{y_2+y_1} + T^{y_3+y_1} + T^{y_1+y_2+y_3}$$

Here the compact toric divisor q is defined by Rescaling this by the first term gives us

$$uF = 1 + T^{y_2} + T^{y_3} + T^q T^{y_2+y_3}.$$

In both of these examples, after rescaling, uF is a function with two kinds of parameters; y_i, the generators of $H_1(F_p)$, and q, the homology of the total space. We frequently write

$$uF_q(y_1, \ldots, y_n)$$

to denote these separate types of variables. Due to the fact that we rescale by u, the function is only dependent on $(y_2, \ldots, y_n)$. As a result, in the 3-dimensional case, the function $uF_q = 0$ defines a curve in $\mathbb{C}^2 = \{y_2, y_3\}$. The resulting curve is called the *mirror curve* to X.

Claim 8. *The mirror curve uF_q^ω is a thickening of the pq-web.*

Proof. More precisely: look at the zero set of uF_q^ω. The limiting zero set is called a *amoeba,* and the limit as $\omega \to \infty$ is called the *tropicalization* of uF_q^ω. It is a general fact that this is the tropical zero of the tropical polynomial uF_q^ω, and the tropical zero is dual to the newton polytope defined by its coefficients. Since the newton polytope is given by the toric fan, it follows that the tropical zero set of uF_q^ω is the pq-web. $\square$

3 Contact Geometry

Definition 7. *A contact manifold is a pair (M, α) where*

- *M is a 2k+1 manifold*
- *α is a 1 form satisfying $\alpha \wedge (d\alpha)^k \neq 0$.*

Given a fixed contact form, a Reeb vector field *R_α is the unique vector field satisfying*

$$d\alpha(R_\alpha, -) = 0$$
$$\alpha(R_\alpha) = 1$$

A good class of contact manifolds to keep in mind for examples come from unit cotangent bundles.

Example 7. *Let N be a k dimensional manifold, and fix a Riemannian metric on N. Let S^*N be the* unit cotangent bundle*, consisting of all differential one forms with norm 1. Then we can define the canonical contact form at a point (x, λ) by*

$$\alpha_{(x,\lambda)} = \lambda$$

which is a contact form. Notice here that the Reeb vector flow R_α generates the geodesic flow of the metric on the unit cotangent bundle.

The contact analogue of a Lagrangian submanifolds is a *Legendrian* submanifold.

Definition 8. *Let (M, α) be a contact manifold. A* Legendrian *$\Lambda \subset M$ is a k-dimensional submanifold such that $\alpha|_\Lambda = 0$.*

We should think of these as the natural counterparts to Lagrangian submanifolds of a symplectic submanifold. In fact, there is nice way to take the contact geometry story and upgrade it to a symplectic one.

Definition 9. *Let (M, α) be a contact submanifold. The* symplectization *of M is the symplectic manifold $(M \times \mathbb{R}, \omega)$, where $\omega = d(e^t\lambda)$. To every Legendrian Λ of M, we get a Lagrangian by taking $\Lambda \times \mathbb{R}$.*

This leads us to the idea of Legendrian Contact Homology: to study the $\Lambda \subset M$ a Legendrian submanifold (or maybe its Legendrian isotopy class,) we might be able to study invariants of $\Lambda \times \mathbb{R}$ (like Lagrangian Floer Theory.) While the technical framework of this theory is fairly difficult, our goal today will be to understand the construction and some applications of this tool. Notice that the symplectization of a contact manifold is not compact.

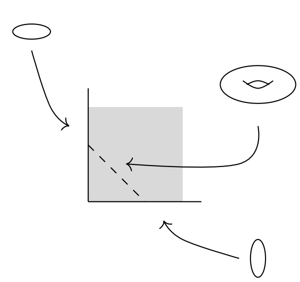
Definition 10. *We say that X is a symplectic filling of M if there exists a compact set $K \subset X$ so that $X \setminus K$ is symplectomorphic to M.*

Here is the key example that we'll be looking at

Claim 9. *Let X be a Calabi-Yau 3-fold. This is a symplectic filling of the contact manifold given by a hyperplane slice of the moment polytope.*

Proof. One thing that we'll need to check is that the preimage of the slice in the smooth submanifold is smooth. We need to show that the projection map is regular onto the slice. Around each of the boundary components, we locally have the same model. ∎

Example 8. *In the case of $\mathbb{C}^2$, the slice that we take of the moment map runs parallel to the line $x + y = 0$.*



Here, one can see that above this slice, we have a torus which is "filled" along it's meridional and longitudinal disks, which shows that the corresponding contact manifold if S^3. This is congruent with the standard definition of the contact structure on S^3 coming from $\mathbb{C}^2$.

This example suggests that we can easily understand the topology of these contact manifolds based on the topology of the slice. By taking a projection of the moment polytope to an axis, we get a Morse-Bott decomposition of the contact manifold, with one S^1 Morse-Bott singularity for every corner of the slice. The gluing of these Morse-Bott singularities is still dependent on the angle on the boundary of the moment polytope enters the slice.

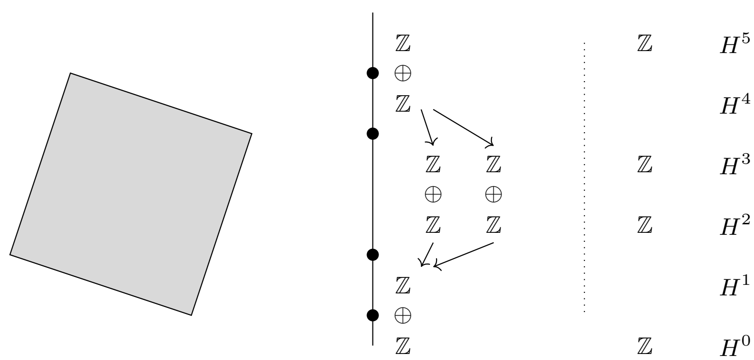
Claim 10. *The data of the contact boundary is given by a polytope with rational slope, where each face is labeled with a integer valued normal vector. We call such information a "Toric Contact polytope."*

Claim 11. *From the data of a Toric Contact polytope, we can compute the homology of $\partial_\infty X$.*

The way that we compute this homology is by using Morse-Bott theory. Each boundary point of the Toric Contact polytope corresponds to a Morse-Bott singularity of the projection to an axis. The index of each critical point will be given by the number of upward facing facets. Each critical submanifold has an upward and downward flow manifold, which gives us a subtorus of $\mu^{-1}(p)$. By taking the intersection number between the upward and downward subtorus corresponding to each critical point, we count the number of flow trajectories between critical submanifolds, giving us the differential on the spectral sequence computing the homology.

Proof. We compute the homology by using the Morse-Bott spectral sequence. $\square$

Example 9. *For example, in the Resolved Conifold, the "slice" looks like a square, and the homology can be computed using the Morse-Bott Spectral sequence:*



We get that the homology matches that of $S^2 \times S^3$, which is actually what this contact manifold is. By using a conifold transition, one can exhibit this as the ideal contact boundary at infinity of T^*S^3 if desired.

Suppose $(\partial_\infty X, \alpha)$ is a contact manifold, and (X, ω) a symplectic filling of ∂_∞. Given a Legendrian $\Lambda \subset \partial_\infty X$, we can ask what Lagrangians L have Λ as a limiting boundary at infinity. These Lagrangians (under suitable compatibility conditions with the symplectization) are called *Lagrangian Fillings of* Λ. This is how we will approach our method to studying the moduli space of Aganagic-Vafa branes. Each Aganagic Vafa-brane corresponds to a filling of the same Legendrian at ∞. By studying this Legendrian without looking at the filling, perhaps we can get a handle on what kind of Lagrangian fillings (like AV-branes) there are.

4 Legendrian Contact Homology

Legendrian Contact Homology should perhaps be interpreted as a contact-version of Lagrangian Intersection Floer homology. There is a dictionary between Legendrian Contact Homology of Λ and the Wrapped Floer homology of $\mathbb{R} \times \Lambda$ that we will not explore here, but can be a useful place for intuition if you are more familiar with one or the other theory.
One first-level interpretation of Floer theories is that they allow us to "rigidize" homotopies between

topological structures. As an non-rigorous analogy: imagine that one wanted to study the set of loops on X up to the relation of homotopy. This is some infinite dimensional space, and the set of homotopy relations is also infinite dimensional, so there isn't a good way to get a geometric handle on this problem. However, if X is endowed with some additional geometry, one could instead study the minimal length loops on X up to area-minimizing homotopies.
The formulation as stated above doesn't really work, but it provides us with a rough idea on how to go about studying some topological objects. In this case, we'll want to study the chords from Λ to itself. The idea of an *energy minimizing* chord is encapsulated in the geometry of the Reeb flow, and the notion of a *flow line for the energy* is given studying the space of pseudoholomorphic curves.

Definition 11. *Let Λ be a Legendrian in (M, α). A* Reeb Chord *of Λ is a flow line of the Reeb vector field R_α which starts and ends on Λ.*

In this interpretation, the Reeb chords are like critical points of the energy function

$$\mathcal{A} : \{ \text{ Chords } \gamma \text{ of } \Lambda\} \to \mathbb{R}$$
$$\gamma \mapsto \int_\gamma \alpha$$

Example 10. *In the case of the cotangent bundle, the Reeb chords of a Legendrian given by a conormal sphere are exactly the perpendicular geodesics.*

Here, it doesn't look like we're using the metric to make a definition, but in fact we use it to define the cosphere bundle.

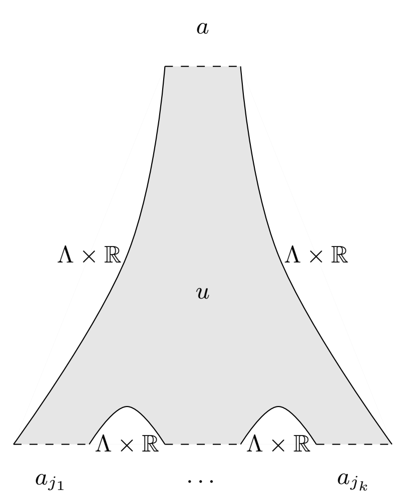
Definition 12. *A* compatible almost complex structure *J for $M \times \mathbb{R}_s$ is one chosen so that $J\partial_s = R_\alpha$.*

Definition 13. *Let Σ be a Riemann surface with conformal structure J, possibly with punctures and boundary components. We say that a map $u : \Sigma \to X$ is* pseudoholomorphic *if*

$$df \circ j = J \circ df.$$

With favorable conditions, the moduli space of such curves in a fixed homology class has a compactification by broken curves.

Definition 14. *Let $a, a_1, \ldots, a_k$ be Reeb chords of Λ. Let β be a homology class in $H_2(M, \Lambda)$. Let D^1_k be a disk with one input boundary puncture and k output boundary punctures. Define the moduli space $\mathcal{M}_\beta(a; a_1, \ldots, a_k)$ to be the set of maps $u : D^1_k \to M \times \mathbb{R}$ with the following boundary configuration*



where the class of u in the relative homology is β.

The holomorphic condition guarantees that curve u lowers the amount of energy of the Reeb chords.

Claim 12. *The moduli space $\mathcal{M}_\beta(a; a_J)$ is compactified by broken curves.*

Definition 15. *Let (M, α) be a contact manfiold, and Λ a Legendrian submanifold of M.*

1. *The* Algebra *associated to the pair* (M, Λ) *will be defined over the ring* $R = \mathbb{Z}[H_2(V, \Lambda)]$*, which is the group ring of the relative homology.*
   The algebra associated to Λ *will encode some of the contact geometry of* Λ*. Let* $a_1, \dots a_k$ *be the set of self- Reeb chords of* Λ*. Then*
   $$\mathcal{A} := R\langle a_1, \dots a_k\rangle$$
   the free non-commutative algebra generated by the Reeb chords. We can turn this into a graded algebra *by giving each of the generators* a_i *an index; for us,* $|a_i|$ *will be taken to be the Conley Zehnder index minus 1. The index of an element in the ring* R *is defined to be* 0.

2. *The* differential *of this algebra will be given by a count of holomorphic strips in the symplectization.*
   $$\partial(a_i) := \sum_{\beta, J} |\mathcal{M}_\beta(a; a_J)| t^\beta a_J.$$

Remark 2. *I'm not actually sure how much of this theory has been fleshed out in full; it is known for instance that this works in the case of contact manifolds arising as jet bundles; however, claims that follow are likely dependent on that.*

Remark 3. *For those who have seen Floer theory before: why is this so much more complicated?*

- *The inclusion of Reeb chords of index 0 means that a disk with k outputs can bubble off an additional output. Therefore, in order to have a theory which can capture the boundary configuration as composition of flows, we need to include multiple outputs in our differential, similar to SFT.*
- *An additional complication in the theory is that the algebra* $\mathcal{A}$ *is non-commutative. This difficulty comes from the fact that the order of the boundary components of these disks is important!*

Since this gives us the structure of a non-linear DGA, it can be in practice difficult to compute the Legendrian contact homology of some Legendrian.

One way to simplify this theory is to supply an *augmentation* to this differential graded Algebra.

Definition 16. *An* augmentation to a ring S *of a DGA (over a ring R)* $(\mathcal{A}, \partial)$ *is a graded unital ring homomorphism*
$$\epsilon : \mathcal{A} \to S$$
so that $\epsilon \circ \partial = 0$.

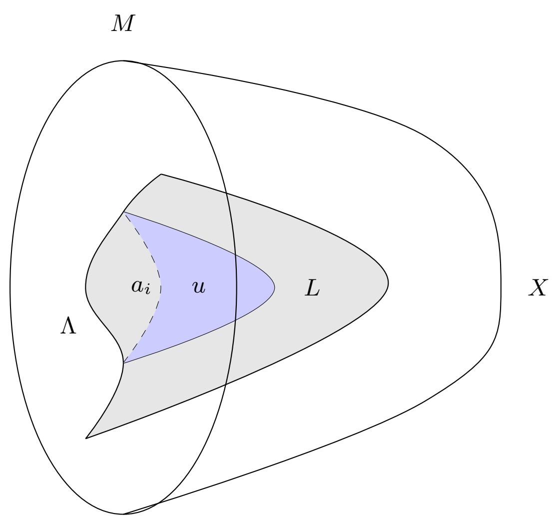
The main difficulty of working with Legendrian contact homology is that the differential is non-linear. A choice of augmentation allows you to construct a linearized version of the homology to work with instead.

Definition 17. *Let* $\epsilon : \mathcal{A} \to S$ *be a augmentation. Suppose that* $\mathcal{A} = R\langle a_1, \dots, a_k\rangle$*. Then we extend* ϵ *to a map*
$$\epsilon_S : S\langle a_1, \dots a_n\rangle \to S$$
Then $C_*^{lin} := (\ker \epsilon s)/\ker(\epsilon s^2)$ *is a finite graded module (generated by the* a_i *with degree not equal to 0) and is graded.*
The map ∂ *descends to a map on* C_*^{lin}*. Notice that now* ∂ *is linear, and it gives us the* linearized homology $H_*^{lin}(M, \Lambda, \epsilon)$

It is not the case that a linearization always exists! Here is maybe an intuition for what a linearization does:

- Take the map $\partial : \mathcal{A} \to \mathcal{A}$, and split it as a sum $\partial = \sum_{l=0}^\infty \partial_l$, where l is word length of $\partial(a)$ in the generators of $\mathcal{A}$.
- In particular if ∂_0, which maps $\mathcal{A} \to R$ is zero, then we have that $\partial_1^2 = 0$, so the restriction to the *module*
  $$\partial_1 : R\{a_1, \dots a_k\} \to R\{a_1, \dots a_k\}$$
  is now a chain complex over R. This would be very good.
- Unfortunately this is rarely the case. You can hope to fix this by adding in a *augmentation.* If you are familiar with the language of A_∞ algebras, you should think of this as finding μ_0 term that turns our DGA into an actual chain complex when the A_∞ algebra is weakly unobstructed. This can give us some intuition for a geometric origin for constructing augmentations.

Based on the previous discussion, we have the following geometric intuition for constructing an augmentation for a Legendrian contact homology. Let X be a symplectic filling for M, that is, a symplectic manifold with contact boundary given by M. Furthermore, let's pick an exact Lagrangian L inside X so that $\partial L = \Lambda$.
To each a_i, we can now associate a class in $H_2(X, L)$, given by the count of holomorphic disks u with boundary in L, and one puncture limiting to a_i.



This defines a map from $\mathcal{A} \to Z[H_2(X, L)]$ which on generators gives this count, and on coefficients is the usual map between $H_2(M, \Lambda) \to H_2(X, L)$. One can check that this is an augmentation in a similar way to checking that $\partial^2 = 0$.

Remark 4. *By picking different augmentations of Legendrian contact homology, we can construct invariants of Legendrians. There is another interesting invariant that we could study– which is what kinds of Augmentations can we attach to the Legendrian DGA.*

As an augmentation is a purely algebraic piece of data, the set of constraints that define an augmentation give us an algebraic variety, called the *Augmentation Variety* of $(\mathcal{A}, \partial)$. We denote this as $V_{aug}(\mathcal{A}, \partial)$.

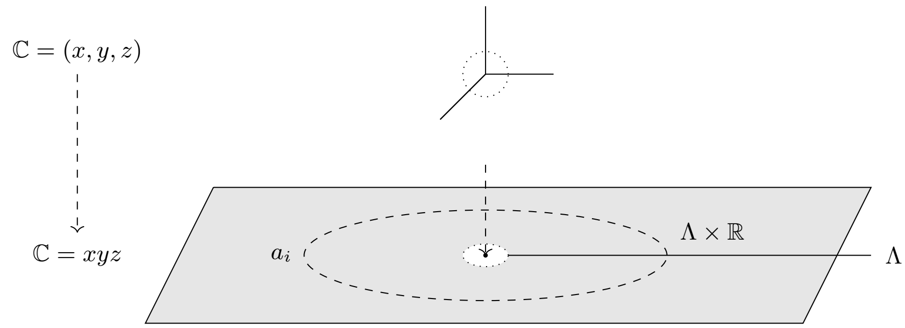
5 A computation of Legendrian Contact homology

In this section, we bring together these stories. Let X be a toric variety, with $\partial_\infty X$ contact boundary at ∞. Let Λ be the product Legendrian in the boundary given by the end of the Aganagic Vafa Brane.

Claim 13. *The Legendrian contact homology of Λ is given by the combinatorics of the fan.*

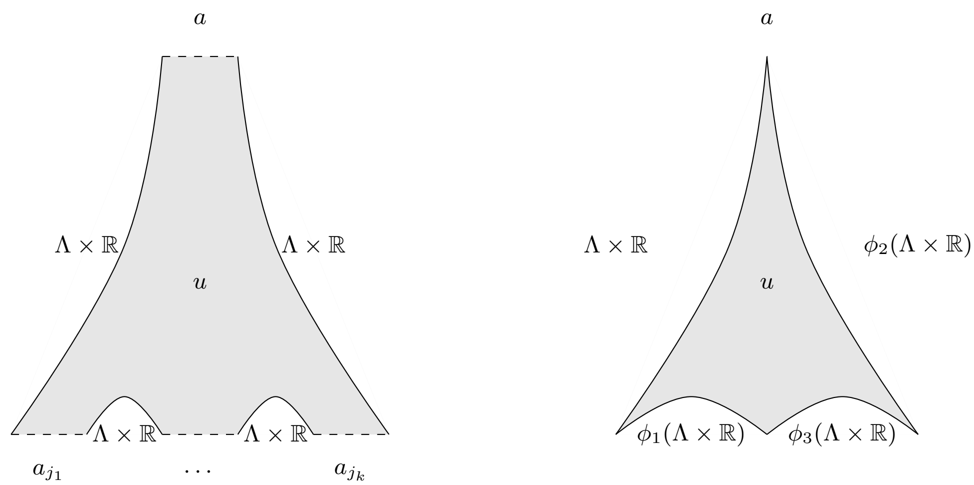
We use the following intuition:

- For appropriately chosen product torus, the reeb flow of the torus is given by the flow of χ_w. The Reeb chords have index given by the number of times they wind around χ_w. In each degreee, the number of Reeb chords is given by the generators of $H_*(T^n)$. Let $a_1, \ldots, a_n$ be generators of $H_1(T^n)$. Then the chord a_I^k is suppose to be the chord corresponding to the homology class $\bigwedge_{i \in I} a_i$, which winds around k times.

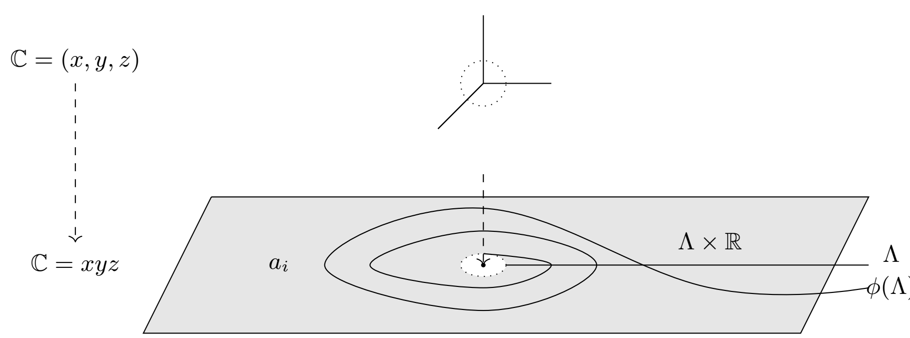


From before, the coefficient ring that we work over will be $R[[H_2(X, F_p)]]$. The Legendrian contact homology is a differential graded algebra, with algebra $\mathcal{A} = R\langle \alpha_I^k \rangle_{I,k}$.

- The count of disks with appropriately marked boundary components is invariant under hamiltonian isotopy. By applying Hamiltonian isotopy, we can instead count intersection points of $L, \phi(L)$, where $\phi(L)$ is an appropriate Hamiltonian isotopy.



The differential of Legendrian Contact Homology becomes the differential in Lagrangian Intersection theory.



This is now a computation of lagrangian floer homology with clean intersections. Let us first focus on the example where we have 2 intersections. The differential splits into 2 portions based on the projection of χ_w; a portion in the fiber of χ_w and a projection in the base. The strips counted in the fiberwise portion give us a differential on the $C^\bullet(F_p)$ correspond to the morse differential. The differentials which project to strips in the base correspond to disks in the product torus. Therefore, the index-1 portion of the differential in Legendrian Contact homology is

$$\partial(a_I^k) = \hat{F}(y_1, \ldots, y_k)a_I^{k-1}$$

where $\hat{F}_q(y_1, \ldots, y_k)$ is the polynomial corresponding to the *non-compact* toric divisors of X.

Example 11. *In the resolved conifold, the differential is given by*

$$\partial(a_I^k) = ((T^{y_1} + T^{y_1+y_2} + T^{y_1+y_3} + T^{q+y_1+y_2+y_3})a_I^{k-1}$$

An augmentation *induced by a filling specializes us to the ground ring. Here q is the homology class of the compact $\mathbb{CP}^1$ in the resolved conifold.*

$$\partial_{fil}(a_I^k) = ((T^{y_1} + T^{y_1+y_2} + T^{y_1+y_3} + T^{q+y_1+y_2+y_3})a_I^{k-1}$$

The topology of the filling tells us that certain homology classes will be killed off by the map between coefficient fields.

$$\partial_{fil}(a_I^k) = (1 + T^{y_2} + T^{y_3} + T^{q+y_2+y_3}).$$

In order for this to be an augmentation, we need this differential to be zero. Accordingly, we've cut out the augmentation variety.
Notice here that the topology of the filling is given to us by the choice of character χ_w.

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