0.1 Stability Structures

The goal of these notes are to explore the geometric intuition between stability, and notions of Stability in the Fukaya category.
The primary references for these notes were,

- The main ideas of these notes come from Bridgeland's paper on stability conditions for triangulated categories [Bri07]
- This talk is a continuation of last week's talk on [HKK14], relating geometry of a surface to its Fukaya category.
- I mostly used [Tha96] to learn about the classical versions of stability conditions. The more algebraic framework for stability conditions can be found in [HL10] and [Mum77] (where many of these ideas were developed.)
- The example of stability conditions for $D_b \operatorname{Coh}(\mathbb{CP}^1)$ is an summary of the notes [Sia].

0.1.1 Geometric Motivation for Stability

The classic stability problem involve the study of vector bundles (or sheaves) over an algebraic surface. We should specify what types of bundle we will try to classify (and up to what type of equivalence...)

Example 0.1.1

For example, one might try to classify the *complex vector bundles* over Σ up to complex bundle isomorphism. This turns into a purely topological problem. Every vector bundle is completely classified by its *rank*, and it's first Chern class (which we will call the *degree* of the vector bundle. One can prove this by checking that the obstruction to finding a vector bundle isomorphism is given by the Chern class.

This indicates that complex vector bundle isomorphism identify too many objects to make this problem an interesting one. The next harder problem that one might ask is to classify the *holomorphic vector bundles* over Σ. As every holomorphic vector bundle is a complex one, we should first break up this problem by rank and degree. So, our question becomes :

Question 0.1.2 Is there a suitable way to describe the holomorphic vector bundles of degree d and rank k?

There are many such non-isomorphic holomorphic bundles of fixed rank and degree.

Example 0.1.3

Let's set $d = 0$ and $k = 1$– so we are looking at degree 0 line bundles over our surface. The moduli space of these objects are determined by the *Jacobian Variety* of Σ, which is the torus $H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z})$. In this case, we have a nice moduli space of bundles.

It is not always the case that we will be able to give this moduli space a nice structure.

Example 0.1.4 [Tha96]

We will construct a 1 dimensional family of vector bundles $E_t \to \Sigma$ so that E_t are all isomorphic, for $t \neq 0$.
We will construct a bundle over $C \times \Sigma$. Take L a line bundle with positive degree, so that $H^1(X, L^{-1}) \neq 0$. Then, let's identify $\mathbb{C}$ with a 1-dimensional subspace of $H^1(X, L^{-1})$. Then every $t \in \mathbb{C}$ determines an extension

$$X \to E_t \to L$$

which are parameterized by $H^1(X, L^{-1})$. In fact, these fit into a bundle over $\mathbb{C} \times \Sigma$, (which you can see by considering the Künneth formula $Ext(\mathbb{C} \times X, L) = H^1(X, L^{-1}) \otimes H^0(C, \mathcal{O})$ whose restriction to each $\{t\} \times \Sigma$ is exactly E_t.
When $t = 0$, we are taking a trivial extension, so $E_0 = \mathcal{O} \oplus \mathcal{L}$. However, the extension chosen non-zero

t may be non-isomorphic (and will necessarily have the same degree and rank.) If there were a moduli space of rank 2 vector bundles, we would have that E_0 is in the closure of E_t, so a moduli space of rank 2 vector bundles (with fixed degree) will be non-Hausdorff.

Here are two different approaches to solve this problem:

- You could develop a *moduli stack*, which encodes this point E_0 as some kind of "stacky-point." This is not the approach that we will take here.
- You could throw out the point E_0 from the objects that you are trying to describe in your moduli space. This will be our approach.

0.1.2 Stable Bundles

From here, when we say bundle we mean *holomorphic vector bundle*

Definition 0.1.5 A bundle E is called *semi-stable* if for every proper subbundle $F \subset E$ we have

$$\frac{\deg F}{\operatorname{rank} F} \leq \frac{\deg E}{\operatorname{rank} E}$$

If we have strict inequality, we call the bundle *stable*

While I won't get into the details here, the notions of stability can be extended to sheaves in many cases. We will call $\frac{\deg F}{\operatorname{rank} F} = \sigma(F)$ the *slope* of the vector bundle. A subbundle $F \subset E$ is called *destabilizing* if $\sigma F \geq \sigma E$. To every vector bundle E, we can associate the *Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

so that each E_i is the maximal rank destabilizing bundle of E_{i+1}. This is a bit akin to looking at a group G and filtering it by largest normal subgroup (the Jordan-Hölder filtration.) There is an associated graded sheaf with graded components $G_i = E_i/E_{i-1}$, where each graded component is a stable sheaf.
The primary usefulness of stability is that it limits the decomposition of your bundle. In particular, every stable vector bundle is also simple one.

Lemma 0.1.6 Suppose that E is a stable bundle. Then if $F \to E \to Q$ is a short exact sequence of sheaves, $\sigma(E) < \sigma(Q)$

Corollary 0.1.7 Suppose that E is a stable bundle. Then E is not a trivial extension of 2 bundles.

Proof. Suppose for contradiction that $E = F_1 \oplus F_2$. Then F_1 is both a quotient and a subbundle of E, so the slope of F_1 must be both greater and less than the slope of E. ■

Corollary 0.1.8 Suppose that E, E' are both stable bundles of the same rank and degree. Then $\hom(E, E')$ is either $\mathbb{C}$ or 0.

Proof. First, we show that ever map is either trivial or an isomorphism. Let $f : E \to E'$. Then let K and I be the smallest subbundles containing the kernel and images of f respectively. Then $I \oplus K$ is a sheaf of the same rank and degree as E or E'. However,

$$\sigma(I) < \sigma(E) = \sigma(I \oplus K)$$

$$\sigma(K) < \sigma(E) = \sigma(I \oplus K)$$

so I and K are not proper subsheaves, so the map f is an isomorphism, or it is the trivial map.
Whenever $\hom(E, E')$ is either entirely isomorphisms or trivial, then it is isomorphic to $\mathbb{C}$. ■

In particular, whenever E is stable, $\mathrm{End}(E) = \mathbb{C}$, and E is called *simple*.

Theorem 0.1.9 [Mum77] The Moduli space of stable vector bundles over Σ of fixed rank and degree is a projective variety.

This theory generalizes outside the context of surfaces and vector bundles.

- In general, this theory applies to torsion-free sheaves on projective varieties with choice of ample line bundle $\mathcal{O}(1)$. You must replace the degree of the vector bundle with the *Hilbert polynomial*

$$P_E(m) := \chi(E \otimes O(m)) = \sum (-1)^i h^i(X, E \otimes O(m))$$

  where inequalities between the slope of a sheaf are determined by the lexicographical order on polynomials.

- If X is Kähler, we can work with a slightly easier definition of slope, by taking ω a Kähler form on X, and defining the *degree* of a vector bundle to be

$$\int_X c_1(E) \wedge \omega^{n-1}.$$

- In this context, we can also come up with a moduli space of stable sheaves.
- Likewise, the Harder-Narasimhan filtration extends to coherent sheaves.

0.2 Extending to categories

We would now like to extend this notion to the derived category of coherent sheaves, and find an axiomatization of this structure.

1. Start with an object $E^\bullet$ in the derived category. The n- *truncation* of $E^\bullet$ replaces is

$$\tau^n(E) := 0 \to (E_n / \operatorname{Im}(d_{n-1})) \to E_{n+1} \to \cdot$$

   which has the same cohomology of as E for $i \geq n$. These f

2. Associated to an object $E_\bullet$ we have a sequence of exact triangles

$$\begin{tikzcd}
0 = \tau^{-l}E_0 \arrow[r] & \tau^{-l+1}E \arrow[r] \arrow[d] & \tau^{-l+2}(E) \arrow[r] \arrow[d] & \cdots \arrow[r] & \tau^{i}E \arrow[r] & \tau^{i+1}E = E \arrow[d] \\
& A^{-l} \arrow[ul, "{[1]}"] & A^{-l+1} \arrow[ul, "{[1]}"] & & & A^{i+1} \arrow[ul, "{[1]}"]
\end{tikzcd}$$

   where A^{i+1} is complex with one group $H_{i+1}(E)$ concentrated at one place. So, these A^i are all coherent sheaves over X.

3. To each A^i we now have a Harder-Narasimhan filtration. Let's denote this filtration A^i_k, and the graded parts G^i_k

4. By using the axioms of a triangulated category we can assemble these together into the following diagram:

$$\begin{tikzcd}
0 = E^{-l}_1 \arrow[r] & E^{-l}_2 \arrow[r] \arrow[d] & E^{-l}_3 \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{i+1}_{k-1} \arrow[r] & E^{i+1}_k = E \arrow[d] \\
& A^{-l}_2 \arrow[ul, "{[1]}"] & A^{-l+1}_3 \arrow[ul, "{[1]}"] & & & A^{i+1}_k \arrow[ul, "{[1]}"]
\end{tikzcd}$$

   for some choice of E^j_k

Proof. There is an exact triangle

$$A_{k-1} \to A_k \to G_k$$

As our category is triangulated, we can find the dashed morphism in this diagram:

$$\begin{tikzcd}
\tau^i E \arrow[r] \arrow[d, dashed] & \tau^{i+1} E \arrow[r] \arrow[d] & A_k^{i+1} \arrow[d] \\
A_{k-1}^{i+1} \arrow[r] & A_k^{i+1} \arrow[r] & G_k^{i+1}
\end{tikzcd}$$

By extending to a this dashed arrow to an exact triangle (with object E_k^i, and applying the octahedral axiom, we can find the required sequence. ∎

5. The groups G_k^j are the associated Harder-Narasimhan filtration of E. They are uniquely determined.
6. Notice that the groups G_k^j are given by the objects $G_k[-j]$, where these are the chain complexes concentrated at degree $[-j]$. The ordering of the complex is lexicographical in slope of the complex and the degree shift. We can combine these pieces of data by upgrading our notion of "slope" by the function $\phi(G_k^j)$ which is defined by

$$e^{2\pi i\phi(G^k[-j])} = -\deg(E) + i\,\mathrm{rank}(E)$$

$$\lfloor \phi(G^k[-j]) \rfloor = -j$$

Then the G_j^k are ordered by ϕ.

It looks like this construction not only relies on the derived category $D_b\,\mathrm{Coh}(X)$, but also a lot on the stability that we had in $\mathrm{Coh}(X)$ itself. The stability structure chosen also depends on the choice of exact triangles that we used in step 2. We would like to eliminate these pieces of data in our construction of a stability condition.

- First, we'll need a way to pick out this function ϕ. Notice that if $A \to B \to C$ is an exact triangle of sheaves, that $\phi(B) = \phi(A) + \phi(B)$. So, we can think of ϕ as coming from a group homomorphism from the Grothendieck group of this triangulated category.

- Once we have such a homomorphism, we will declare some objects of our triangulated category to be the *stable subcategories*
- While the first diagram of the exact triangles that we considered came from considering the t-structure on $D_b\,\mathrm{Coh}(X)$, the final sequence of triangles we considered did not come from this structure. So, our filtration result will be something about the existence of a such a sequence of triangles.

Definition 0.2.1

A stability condition on a triangulated category is

- A morphism $Z : K(\mathcal{C}) \to \mathbb{C}$ called the *central charge.*
- For each $\phi \in \mathbb{R}$, a full additive subcategory $\mathcal{C}^\phi$ of *semi-stable objects,*

These should satisfy the following axioms:

(a) $C^\phi[1] = C^{\phi^{i+1}}$

(b) If $\phi_1 > \phi_2$, then $\hom(\mathcal{C}^{\phi_1}, \mathcal{C}^{\phi_2}) = 0$.

(c) For any semi-stable object E with phase ϕ,

$$Z(\mathrm{cl}(E)) = |Z(\mathrm{cl}(E))| e^{i\phi}$$

(d) For any object E, there is a *Harder-Narasimhan Filtration*

$$\begin{tikzcd} 0 = E_1 \arrow[r] & E_2 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{k-1} \arrow[r] & E_k = E \arrow[d] \\ & A_2 \arrow[ul, "{[1]}"] & A_3 \arrow[ul, "{[1]}"] & & & A_k \arrow[ul, "{[1]}"] \end{tikzcd}$$

where the A_n are of decreasing phase.

Notice that in the case of $D_b\,\mathrm{Coh}(X)$, the morphism from the Grothendieck group to $\mathbb{C}$ is doing something exactly like taking the slope. This ties back with our ideas from before, as the K group of the category should do something like classify the vector bundles over the group (up to some natural relations.)

Definition 0.2.2
t-structure, Heart of a t structure.

Let D be a triangulated category. A *t-structure* on D is a pair of full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ such that

- $\hom(D^{\leq 0}, D^{\geq 0}[1]) = 0$.
- For every object X, there is an embedded triangle of the form
$$X^{\leq 0} \to X \to X^{\geq 0}[1].$$
- $D^{\leq 0}$ and $D^{\geq 0}$ are closed under positive and negative shifts respectively.

The intersection of $D^{\leq 0} \cap D^{\geq 0}$ is called the *heart* of the t-structure, and it is an abelian category.

This is a good candidate for the "abelian category" whose derived category is the original thing.

Lemma 0.2.3

A full additive subcategory of a is the heart of a bounded t-structure if and only

- $\hom(A[i], B[i+n]) = 0$ for all $i \in \mathbb{Z}$, $n \in \mathbb{N}$.
- For every nonzero object $E \in \mathcal{D}$, there is a sequence of triangles with one entry in $\mathcal{A}$.

From this lemma, one can see that

0.2.1 Space of Stability Structures

The space of stability structures comes with a complex structure. The topology of the space is defined by a metric.

Definition 0.2.4

Let $\mathbb{C}$ be a triangulated category, with identification $cl : K_0(\mathcal{C}) \to \Gamma$. Let $\sigma_1, \sigma_2 \in \mathrm{Stab}(\mathcal{C}, \Gamma)$ be 2 stability structures. The distance between these structures is the maximum (over all objects E)

- The phase change of largest phase semi-stable component of E
- The phase change of the smallest semi-stable component of E
- The quantity $\log \frac{\sum |Z_2(A_i)|}{\sum |Z_1(A_i)|}$

The complex structure is shown that the projection of $\mathrm{Stab}(\mathcal{C}, \Gamma)$ to the characters of Γ is a local homeomorphism.

Example 0.2.5

Bundles on $\mathbb{CP}^1$. Variations of Stability Structure, and Wall Crossing. Change of heart of the category

0.3 Mirror Symmetry Predictions and Results

The Homological mirror symmetry conjecture tells us that we should be able to relate structures on the derived category of coherent sheaves to structures on $\mathcal{F}(S)$. In particular

- Bridgeland stability structure on $D_b \operatorname{Coh}(X)$ should give us a stability structure on $\mathcal{F}(S)$.
- The moduli space of stability structures should be related to some moduli space of geometric structures on S.

Conjecturally, on a compact Calabi-Yau M with a choice of holomorphic volume form , the special Lagrangians of M are stable objects, with the central charge defined by the integral of the holomorphic volume form over L. Then the space $\operatorname{Stab}(\mathcal{F}(M))$ is related to the moduli space of these holomorphic volume forms.
At least one of the axioms of the Bridgeland stability criterion is easy to verify in this case. When we you have 2 special lagrangian manifolds, with different phases. This requires you to be precise with your definitions of grading. Recall when we discussed grading of morphisms on the topological Fukaya category, it was given by computing the intersection index of the two arcs. When these arcs are special lagrangian (i.e. they are geodesics) then the intersection index is either 0 or 1, dependent on which one has greater phase.

Theorem 0.3.1 In the case of the topological Fukaya category of a surface, a flat structure determines a Bridgeland Stability Condition.

Outline of proof. The part of the proof that I want to outline is the construction of the Narasimhan-Harder Filtration.
It suffice to check this for indecomposable objects E, which we know are exactly represented by a graded admissible curves with indecomposable local systems. We also know that these are represented by a twisted complex of geodesics. Let $\phi_1 > \phi_2 >\leq> \phi_l$ be the phases of these geodesics. We can define a semi-stable object B_i which is the twisted complex of those geodesics α_i with phase ϕ_i. The twisted complex has non-zero coefficient if two geodesics pass through a component without changing phase. ∎

0.3.1 Moduli of Flat Structures

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