1 Sheaf-Like approaches to the Fukaya Category

The goal of this talk is to take a look at sheaf-like approaches to constructing the Fukaya category of a surface. We should expect that there is some kind of sheaf-like structure to the Fukaya category, as every point is contained in a neighborhood which is symplectomorphic to a neighborhood of the origin of $\mathbb{C}^n$ equipped with the standard symplectic structure. If there was a way to treat the Fukaya category as a sheaf, we could "locally" compute the Fukaya category on small neighborhoods, and then glue together the category from those parts.
It was conjectured by Kontsevich that the Fukaya category of a Stein manifold can locally be computed on a Lagrangian skeleton [Kon09]. We'll take a look at some of the progress that has been made on the simplest case of Stein manifolds, which are surfaces with punctures.

We'll be mostly following the work in [HKK14], but this exposition is also influenced from the ideas of [STZ14], which also computes a Fukaya-like category for punctured surfaces. There are several different approaches to computing the Fukaya category for surfaces (or using sheaf like techniques) which I should mention here:

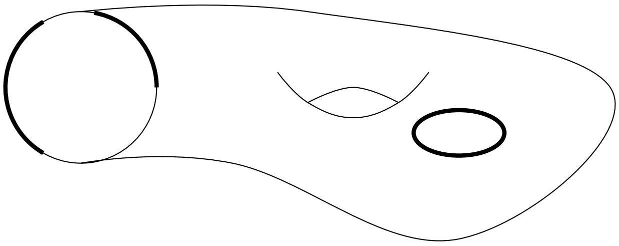
- One thing that we need to understand before computing the Fukaya category in a sheaf like way on the Lagrangian skeleton is the Fukaya category of the cotangent bundle of a Lagrangian. These make the basic pieces of the skeleton that we glue together. One approach to this is to use microlocal sheaves, as in [NZ09]. If we can find a sheaf-like description for the Fukaya category, we would essentially know that the Fukaya category for Stein manifolds can be computed using the microlocal geometry.
  A different approach to computing the Fukaya category of the cotangent bundle is to use the wrapped Fukaya category. This has been computed by Abouzaid, who shows that the triangulated envelope of the wrapped Fukaya category of T^*L is generated by a single cotangent fiber, and is quasi-isomorphic to the category of twisted complexes on chains of the based loop space [Abo11].
  For the case of wrapped Fukaya categories on punctured surfaces, the basic "neighborhood" is the sphere with punctures. The Fukaya category for $S^2 \setminus D$ is computed in [AAEKO13], providing a building block for computing the Fukaya category of punctured surfaces.
- After describing the "building blocks" of the Fukaya category, we need some way of gluing them together. Sibilla, Truemann, and Zaslow very explicitly construct a Fukaya-like category for punctured surfaces in their paper on Ribbon graphs. Here, the ribbon graph is suppose to be a combinatorial representation of a plumbing.
  A more general goal would be to understand the Fukaya category of the plumbing $T^*L_1 \# T^*L_2$ in terms of the topology of L_1 and L_2. Progress has been made on this front in the paper [Abo09]. In the case of wrapped categories and surfaces, Heather Lee's thesis [Lee15] provides instructions on how to glue together a Fukaya category using the building blocks of [AAEKO13], allowing us to explicitly compute the wrapped Fukaya category for these spaces.

1.1 Surfaces and Arcs

We'll first describe a version of the Fukaya category for punctured surfaces, called the *topological Fukaya category.*

Definition 1.1 A *marked surface* S is a surface with corners, along with a collection $M \subset \partial_1 S$ so that each point in $\partial_0 S$ belongs to exactly on connected component of M, and the $\partial M = \partial_0 S$.

Pictorially, a marked surface is a surface with corners whose boundary components have been broken into an alternating collection.

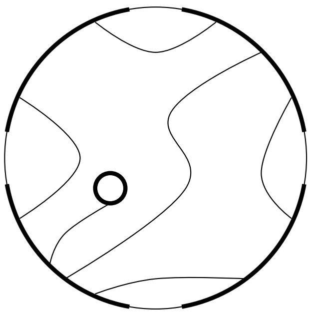


We can break the surface up into smaller portions by adding in arcs which split the surface into contractible components.

Definition 1.2 Let S be a marked surface. An *arc* of S is a embedded curve α so that

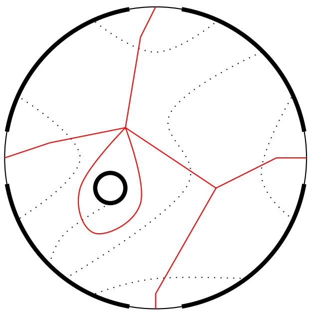
- $\partial\alpha \subset M$ and α is transverse to M.
- $\partial\alpha$ is not path isotopic to a subset of M.

A *arc system* is a collection $A = \{\alpha_i\}$ of arcs, which are pairwise disjoint and non-isotopic. It is called a *full arc system* if it gives a polygonal decomposition of the surface.
A *boundary arc* is an arc isotopic to a boundary component not contained in M.



A full arc system with 2 internal arcs and 4 boundary arcs

Notice that the dual graph G to a full arc system is a ribbon graph for the surface.



The Ribbon graph dual to our arc system

1.2 Grading of Arcs

If we are interested in modeling the Fukaya category, all of our objects should have some kind of grading structure on them. They get this grading from a choice of foliation on the surface S.

Definition 1.3

A *grading* for S is the choice of a 1-foliation for S, that is a section of $\eta : T \to \mathbb{P}(TX)$.

Given any grading of S, we can assign a number to each half-edge h of G by taking the winding number of the foliation as we go around from the half edge to its neighbor. We will call this the *degree* of h, and denote it $d(h)$. Notice at each vertex, we have that the sum of all of the degrees is 2 more than the valency of the vertex (just check the winding of the distribution on a path that goes around the vertex one time.)
From this data, we can construct a combinatorial Fukaya-like category.

Definition 1.4

Given two graded arcs α and β, a *boundary path* between them is a path in M between $\alpha \cap M$ and $\beta \cap M$. If the arcs α and β are graded, then the *degree* of the boundary path is the grading of the concatenation of α and β.
Suppose that $\alpha_1, \dots \alpha_n$ have the property that they bound a disk, that α_i and α_j intersect a common boundary component if and only if $j = i + 1$, and that α_i are drawn in clockwise order. Then we say that the collection of α_i bound a n-disk sequence.

Definition 1.5
Arc Categry

Let (S, M, η) be a marked surface, and A a system of graded arcs on S. Then the A_∞ *arc category* $\mathcal{F}_A(S)$ is defined to have objects A, morphism spaces generated boundary paths, composition by path composition, and higher operations given by counting n-disk sequences.

Here, it $b_0, b_1, \dots b_k$ is a set of boundary points in a disk sequence, we say that $\mu^k(b_k, \dots, b_1 b_0) = (-1)^{d(b_0)} b$.

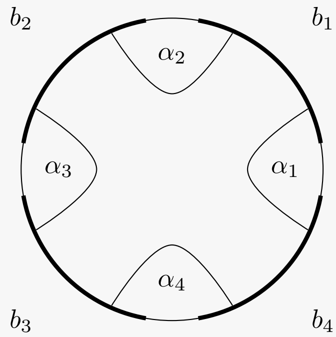
Proposition 1.6
[hkk]

$\mathcal{F}_A(S)$ is a strictly unital A_∞ category.

Example 1.7
Disk with n components

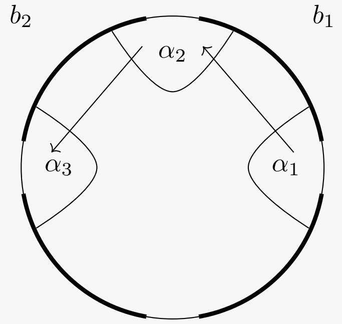
Here, we only have the arcs which cover the other boundary components. Let's index these objects by $\mathbb{Z}/k\mathbb{Z}$.

The only morphisms in this category are between the object α_i and α_{i+1}. There is a single disk in this category, connecting all of the morphisms together.



This means that the twisted complex (given by a chain of the arcs, connected by the morphisms along the boundary) of the first $n - 1$ objects is equivalent to the last object as a twisted complex.
This means that you only need $n - 1$ arcs to generate the triangulated envelope of the Fukaya category of the disk with n boundary components. This means that the arc system $A \setminus \alpha$ has a triangulated envelope equivalent to A. The category $A \setminus \alpha$ is easy to describe: it equivalent to the linear path category of the A_n

quiver.



The A_2 quiver

In some sense, the choice of arc family A does not effect the properties of the category.

Theorem 1.8 [hkk] The Morita[1]equivalence class of $F_A(S)$ has no dependence on A.

Proof. Suppose that A and B are two full arc systems which vary only by the addition of a single arc β. Let's look at one of the disks which contains β at the boundary. This disk is a surface with boundary. We know that this can be represented as the twisted complex of the remaining sides of the disk, so this shows that the triangulated envelopes of $\mathcal{F}_A(S)$ and $\mathcal{F}_B(S)$ are the same.
Now, we have provided a map from the partially ordered set of arc systems to equivalence classes of A_∞ categories. There is a general result that shows that the classifying space of these arc systems is contractible, so we now know that all of the full arc systems lie in one equivalence class. ∎

Definition 1.9 We define *the topological Fukaya category* $\mathcal{F}(S)$ to be the category of twisted complexes over $\mathcal{F}_A(S)$.

By the result above, the topological Fukaya category is independent of choice of full arc system.

1.3 Cosheaf of Categories

While this category has a nice combinatorial definition, it is a little difficult to get our hands on the actual structure of this category. In this section we will show that the topological Fukaya category behaves like a cosheaf of categories.
Let (S, M, η, A) be as before. The dual graph to G is a ribbon graph, which is a Lagrangian skeleton for $S \setminus M$.

Definition 1.10 A *cosheaf* of categories $\mathcal{E}$ on G is

- For each vertex v, a category $\mathcal{C}_v$.
- For each edge e, a category $\mathcal{C}_e$.
- For every incidence between e and v, a functor $\mathcal{C}_e \to \mathcal{C}_v$.

The category of global sections $\Gamma(G, \mathcal{E})$ is defined to be homotopy colimit functors $\mathcal{C}_e \to \mathcal{C}_v$.

This definition also works if replace all of the categorical terms with their respective A_∞ constructions.
We can use ribbon graph decomposition to construct a cosheaf geometrically inspired by the Fukaya category.

- To each e, $\mathcal{C}_e$ has a single object, the arc α dual to e. We define $\hom(\alpha, \alpha) = \mathbb{K}$.

[1] I haven't been able to find what exactly a Morita equivalence of A_∞ categories exactly is, but my best guess is that their categories of A_∞ modules are equivalent. This would mean that their triangulated envelopes are the same

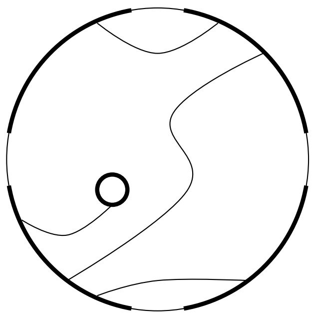
- To each v, we define $\mathcal{C}_v$ to be the category $\mathcal{F}(D)$, where D is the dual disk to the vertex determined by the Arc system. This is the category of the path algebra of the $A_{\deg v - 1}$ quiver.

This forms a cosheaf of categories over G. For each v, e, there are functors from $\mathcal{C}_v \to \mathcal{F}(S)$ which commute with the restriction functors.

Theorem 1.11 [hkk] The sheaf of global sections of this cosheaf of categories is Morita equivalent to the topological Fukaya category.

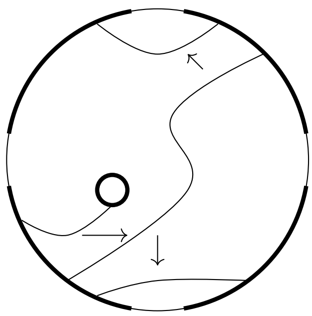
Outline of Proof. The idea is to prove the Morita equivalence by first removing boundary components and reducing to the case where we have a graph with no arcs going to boundary components, then by reducing the ribbon graph by contractions.
Case 1: We first work with a simplified set of generating arcs, which we can find whenever S has at least one boundary arc:

Lemma 1.12 If S has at least one boundary arc, then there is a system of arcs A which cuts S into disks, where each disk is bounded by a boundary arc not belonging to A. We will call this a *full formal system of arcs.*



A full formal system of arcs

Note that a formal system of arcs has no higher multiplications (as we have removed the necessary boundary arcs to close up the disks making the higher multiplication.) In fact, we have that $\mathcal{F}_A(S) = \mathbb{K}Q$ for some graded quiver Q, and set of compatible arrows in Q. Additionally, the category of twisted complexes over a full formal system of arcs is quasi-equivalent to $\mathcal{F}(S)$. This is because the removed boundary arcs are generated by the twisted complex of the remaining arcs.

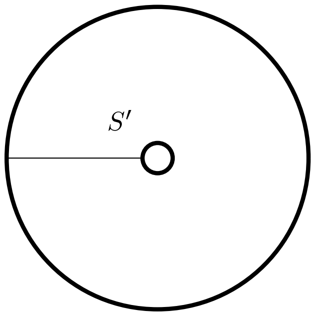


The associated quiver D_3

Note that the act or taking a full system of arcs (which contains a formal full system) and reducing it to the formal full system does not change the associated colimit of cosheaves. In the associated ribbon graph, each vertex has an arc going out not in the image of any incoming edge. We can remove this object as well, and get an equivalent diagram of categories.

- If we have a boundary component, then we have an edge of the ribbon graph going to that boundary arc. When we remove the edge, we are removing an inclusion in our sheaf (so the removal of the corresponding map $\mathcal{C}_e \to \mathcal{C}_v$ will not change our homotopy colimit.
- The resulting diagram has *at each vertex* an object which is not in the image of any edge map. We can remove these objects (without changing the colimit.)We can therefore reduce to the case where we have some colimit of graded linear categories instead.
- One can show that this is the graded linear model of $\mathcal{F}(S)$ (and a proposition shows that this is the homotopy coequalizer.)

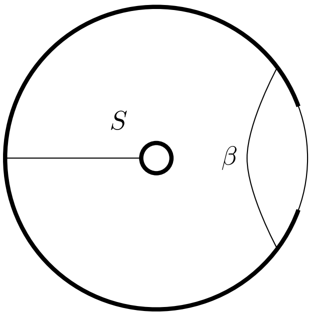
Case 2: So, now we have to work with the case where S has no boundary arcs (that is, the markings on M on S have no boundary.)



There is no formal arc system

In this case, there is a homological algebra computation that shows:

Lemma 1.13 If we modify S to S' by adding in a boundary curve α, then $\mathcal{F}(S) = \mathcal{F}(S')/\beta$, a localization of the topological Fukaya category.



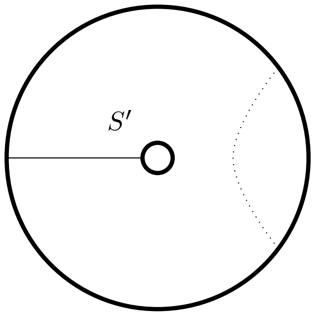
The modified surface

Outline of Localization. The localization of an A_∞ category at an object is given taking the graded vector space of homomorphisms between objects α_1, α_2 and extending them with

$$\hom(\beta, \alpha_2) \otimes \mathbb{K} \otimes \left(\bigoplus_{n \geq 0} (\hom(\beta, \beta) \otimes \mathbb{K})^{\otimes n} \right) \otimes \hom(\alpha_1, \beta)$$

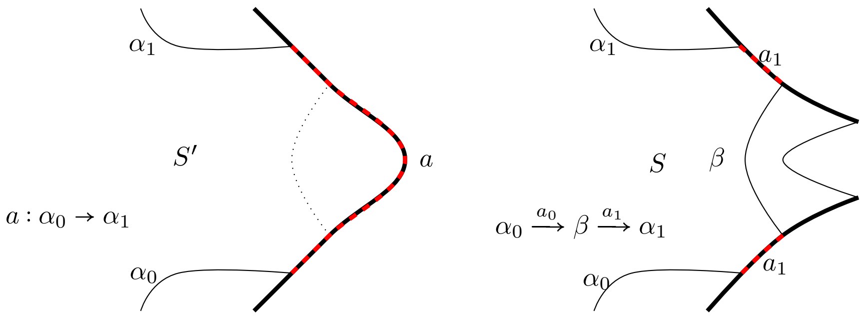
(which you should perhaps visualize as a curve from α_1 to as many curves from β to itself as we need, and then a curve to α_2.

There is no map between $\mathcal{F}(S)$ and $\mathcal{F}(S')$, because the additional arc β has no where to map to. So, we will exted the category $\mathcal{F}(S')$, with an additional "null arc" which is allowed to be null homotopic. The price we have to pay is that our category no longer has $\mu^1 = 0$– we throw this differential in so that the class of the null-arc is zero object in homology.



An additional null-arc

The map from $\mathcal{F}(S) \to \mathcal{F}(S')$ send β to the null arc. The map the other way now has to deal with paths that pass through β, which we write as the composition of the two paths.



Composition of curves in S' becomes broken in S

One can check that once we localize at β, this is an isomorphism of A_∞ categories. The map between $\mathcal{F}_A(S)/\beta$ and $\mathcal{F}'_A(S')$ is a quasi-equivalence of A_∞ categories. ■

This tells us the relation between $\mathcal{F}(S)$ and $\mathcal{F}(S')$, so we need to know the relation between. $\Gamma(G', \mathcal{E})$ and $\Gamma(G, \mathcal{E})$, where G' and G differ by the edge which goes to the "inserted" boundary arc. The replacement is that one of the vertices gets changed from the A_n path algebra to the A_{n-1} path algebra. One can check that this is localization of the category at that path.
There is a homological algebra argument that quotients of categories commutes with colimits, so we actually have the same category. ■

2 Geometrician

Our next goal is to show that the category $\mathcal{F}(S)$ is reasonably geometric. Right now, this is defined as the "triangulated envelope" of some geometrically defined category, but ideally all objects should be represented by geometric curves.

2.1 Admissible Curves

Definition 2.1 Admissible Curves

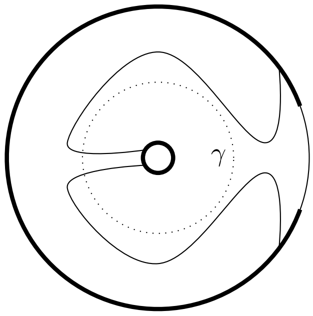
An immersed curve is called *admissible* if

- It bounds no teardrops (unobstructedness, see also Abouzaid's paper [Abo08] on the Fukaya category of higher genus surfaces.)
- If it is a closed curve, it lies in the interior of the surface and is a primitive curve.
- If it is has boundary, the boundary meets the marked points transversely, and it not homotopic to a

marked boundary.

Notice when S is a disk, admissible curves are exactly graded arcs, so the set of admissible curves exactly corresponds to objects of $\mathcal{F}(S)$. We would like to extend this process to all admissible curves on all surface. There are 2 cases to treat: the one where an admissible curve γ is a closed curve, and the one where it is a curve with boundary.

- Let's work with the case of curves with boundary first. In this case, we can take the curve γ in S, and lift it to in an arc in $\tilde{S}$, the universal cover of S. Since our arc is compact, there is some disk $D \subset \tilde{S}$ bounded by arcs chich completely contains the curve γ. The map from $D \to S$ induces a map on the arc systems of D and S, and therefore a map from $\mathcal{F}(D) \to \mathcal{F}(S)$. This qives an object $[\gamma] \in \mathcal{F}(S)$, whose equivalence class is independt of choice of D.
- If the curve γ is a loop, we will break it into individual arcs and reduce to the case above. A loop can always be broken along marked boundary components.



Replacing a loop γ with curve "broken" into arcs.

The loop γ is broken into arcs and curves in the boundary, giving us a twisted complex of admissible curves with boundary. It is therefore a twisted complex of arcs, and describes an object in $\mathcal{F}(S)$. The equivalence class of this object does not depend on the choice of breaking.

Theorem 2.2 If S is a graded marked surface, then there is an equivalence between

- Indecomposable objects in $H^0(\mathcal{F}(S))$
- Admissible curves with indecomposable local systems.

Outline of Proof. The involves classifying representations of objects called nets, and showing that

- All of the indecomposable representations of nets can be understood by looking at maps from nets which are either loops or paths.
- All indecomposable objects in $H^0(\mathcal{F}(S)$ are given by indecomposable representations of a net related to the quiver describing the Fukaya category.
- All nets representing loops or paths are twisted complexes given by a sequence of arcs coming from either a admissible closed curve or admissible curve with boundary.

■

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