Morse Homology

Jeff Hicks

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0.1 Introduction

The idea behind Morse homology is that the height function of a manifold captures a lot of data on the topology of the manifold. More specifically, knowing the critical points of a height function on a manifold gives you information on where the level sets $f^{-1}(t)$ of the manifold (with respect to the height function) change topologically when you cross a critical values of the height function. Similarly, information on the topology of the manifold M, you can get data on what type of height functions are possible to draw on a manifold M. The notes for this section were largely based on [1]. Before we get started, here is some notation that we may use here and in later sections:

0.2 Things about Morse Functions

Definition 0.1

Let $f : M \to \mathbb{R}$ be a smooth function. A *critical point* of f is a point $p \in C$ such that $df_p = 0$— that is all the partial derivatives vanish.

Definition 0.2

Let ∇ be any connection on TM, and p a critical point of the function f. Then define the **Hessian** $H(f,p) : T_pM \to T_p^*M$ as

$$H(f,p)(X) := \nabla_X(df)$$

Lemma 0.3

This definition is independent of choice of connection.

Proof. Given two connections ∇ and ∇', we can check that the difference $\nabla_X(Y) - \nabla_X(Y)'$ is C^∞ linear in both the Y and X arguments, therefore $\nabla_X(Y) - \nabla_X(Y)'$ is a tensor. Therefore the value of $\nabla_X(df) - \nabla'_X(df)$ is only dependent on the value of the value of df. Since we are at a critical point, $df = 0$ and therefore the difference $\nabla_X(df) - \nabla'_X(df) = 0$.
Basically, what this says is that $\nabla_X(df)$ is dependent on the local behavior of df, while $\nabla_X(df) - \nabla'_X(df)$ is only dependent on the value of df at a point p.

Another way to view the Hessian is the matrix of mixed partials at a point, which is stating that the Hessian is a symmetric bilinear on T_pM given by the matrix $H = (\partial^{ij} f)$.

Definition 0.4

Let p be a critical point of the function f. Then we say that p i **nondegenerate** if the Hessian does not have a zero eigenvalue at p. Define the **Morse index** $\operatorname{ind}(p)$ to be the number of negative eigenvalue of the Hessian. A function f is called **Morse** if all of its critical points are nondegenerate. The set of all critical points of index i is denoted $\operatorname{Crit}_i(f)$.

The magics of differential geometry give us the following facts about Morse functions.

Lemma 0.5
Morse Lemma

Let M be a manifold.

1. Functions in $C^\infty(M)$ are generically Morse.
2. Let $f : M \to \mathbb{R}$ be a Morse function. Then the critical points of f are isolated.
3. If p is a nondegenerate critical point of a function of index i, then there exist local coordinates on M around p such that
$$f = f(p) - x_1^2 - \cdots x_i^2 + x_{i+1}^x + \cdots + x_n^2.$$

0.3 Gradient Flows

Recall, if M is a n-dimensional smooth manifold, a **metric** on M is a smoothly varying inner product $g_{ij} : T_pM \times T_pM \to \mathbb{R}$. Another description is that g_{ij} is a symmetric $(0,2)$ tensor field which is positive definite at every p.

Definition 0.6

Given a Riemannian manifold (M, g), the gradient of a function f is the smooth vector field ∇f such that for any vector field X, we have
$$g(\nabla^i f, X) = \partial_i f$$

The gradient of a function can be expressed in local coordinates as follows. Let g_{ik} be the metric tensor in local coordinates. Then we have that
$$\nabla^i f = g^{ik}\partial_k f$$
This is stating that the gradient is obtained by raising the index on the differential $\partial_k f$. Let $f : M \to \mathbb{R}$ be a Morse function, and let V denote the negative gradient of f with respect to g. The flow of the vector field V defines a one parameter group of diffeomorphism $\Psi_s : M \to M$ for $s \in \mathbb{R}$ with $\Psi_0 = \operatorname{id}$ and $\frac{d\Psi_s}{dt} = V$.

Definition 0.7

Let p be a critical point of the function f. Define the **descending manifold**
$$\mathcal{D}(p) := \left\{ x \in M \,\middle|\, \lim_{s \to -\infty} \Psi_s(x) = p \right\}$$
Likewise, define the **ascending manifold**
$$\mathcal{A}(p) := \left\{ x \in M \,\middle|\, \lim_{s \to +\infty} \Psi_s(x) = p \right\}$$

In other words, the descending manifold is the set of all points that have gradient flows that "head away" from the critical point p, and the ascending manifold is the set of all points that have gradient flows that "head toward" a critical point p. In some texts, the terms stable and unstable manifold are used to describe these two objects.

Notice that the index of a critical point gives us a lot of the structure of these manifolds. In particular, the dimension of the descending manifold and codimension of the ascending manifold are equal to the index of the critical point. This is such an important fact, that we will write it out again.
$$\begin{aligned}\dim(\mathcal{D}(p)) &= \operatorname{ind} p\\ \dim(\mathcal{A}(p)) &= n - \operatorname{ind} p\end{aligned}$$

Now we really see where Morse theory is going. If we can understand how the ascending and descending manifolds of critical points interact with each other, we can really get a handle on the structure of the manifold. Our general plan will be to construct a chain complex with groups given by the critical points, and the morphisms determined by the types of flows the occur between different critical points. We will have to impose a small (but important!) restriction on the ascending and descending manifolds to proceed further:

Definition 0.8

Let (M, g) be a Riemannian manifold. A function $f : M \to R$ is called **Morse-Smale** if every ascending manifold is transverse to every descending manifold in M with respect to the metric g. We then refer to (f, g) as a **Morse-Smale pair**.

We won't prove it here, but functions are generically Morse-Smale.
Given a Morse-Smale function, we can talk about the intersections of the ascending and descending manifolds of critical points. These spaces contain all points on flows between p and q. It makes sense to talk about the set of flows lines, which should be one dimension smaller than the set of all points on flows between p and q (as each flow line is one dimensional)

Definition 0.9

Let p and q be critical points. A **flow line** from p to q is a path $\gamma : \mathbb{R} \to M$ such that

- The derivative of γ matches $gradient f$
- γ heads towards p and q, that is $\lim_{s\to-\infty} \gamma(s) = p$ and $\lim_{s\to+\infty} \gamma(s) = q$

We call two flow lines equivalent if they differ by precomposition with translations by $\mathbb{R}$.The set of equivalence classes of flow lines is call the **Moduli space of flow lines from** p **to** q modulo translation, and is denoted $\mathcal{M}(p, q)$.

As suggested before, the modulo space of flow lines can be identified with the intersection of descending and ascending manifolds.

$$\mathcal{M}(p, q) = \mathcal{D}(p) \cap \mathcal{A}(q)/\mathbb{R}$$

The Morse-Smale conditions tell us that $\dim M(p, q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1$. One slightly annoying thing that we have to do now is apply orientations to the moduli spaces. for each p , choose an orientation of the descending manifold. Then we have the isomorphism which determines the orientation of the moduli space uniquely.

$$\begin{aligned} T\mathcal{D}(o) \simeq& T(\mathcal{D}(p) \cap \mathcal{A}(q)) \oplus (TM/T\mathcal{A}(q)) \\ \simeq& T_\gamma \mathcal{M}(p, q) \oplus T_\gamma T_\gamma \oplus T_q \mathcal{D}(q). \end{aligned}$$

Now we have given $\mathcal{M}(p, q)$ some very concrete structure when the index of p and q differ by one. The dimension of $\mathcal{M}(p, q)$ is one dimensional, and the orientation that we associate to it assigns either a + or a − to every point in $\mathcal{M}(p, q)$. in order to count the number of ± points in the moduli space, we will want to know first that there are a finite number of points in the moduli space. This follows from the fact that the moduli space is compact. In fact, even if the index of p and q differ by more than one, $\mathcal{M}(p, q)$ has a natural compactification that is related to the structure of flow lines. The compactification of $\mathcal{M}(p, q)$ is going to be a manifold with corners, where the strata of this manifold are related to flows lines sitting in between p and q

Definition 0.10

A **manifold with corners** is a second countable space N where every point has a neighborhood homeomorphic to $\mathbb{R}^{n-k} \times [0, \infty)^k$, and whose transition maps are smooth.

Theorem 0.11

If (M, g) is a closed Riemannian manifold and f is Morse-Smale, then for any two critical points p, q, the moduli space $\mathcal{M}(p, q)$ has natural compactification to a smooth manifold with corners $\overline{\mathcal{M}(p, q)}$

$$\overline{\mathcal{M}(p, q)} \backslash \mathcal{M}(p, q) = \bigcup_{k \geq 1} \bigcup_{r_1, \ldots, r_k} \mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdot \times \mathcal{M}(r_k, q),$$

This is the theorem that makes all of Morse theory tick. A visualization of the theorem is in Figure 1.

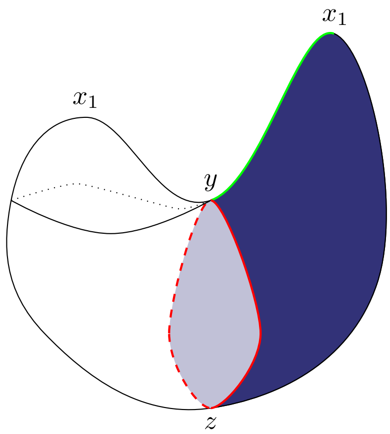

Figure 1: A boomerang shaped spheroid, with some parts filled in. The points marked x_1, x_2, t, z represent the different critical points of the Morse function. The flow space between x_1 and y is represented in green ; x_1 and z is the Orangeregion ;y and z is represented in red. If we look at $\mathcal{M}(x_1, z)$, we see that it is a one dimensional space with open endpoints. In order to compactify these endpoints, we are going to need to add something. What we add is the space $\mathcal{M}(y, z)$, which is two points corresponding to the two flow lines between y and z. The resulting space that we get is $\overline{\mathcal{M}(x_1, z)}$

The flows that correspond to points on the boundary of $\overline{\mathcal{M}(p,q)}$ are called **broken flows**. Each point on the boundary can be represented by a sequence of flows from p to q that may pass through some other critical values.

This theorem is important to us for two reasons. Firstly, if $\mathrm{ind}(p) - \mathrm{ind}(q) = 1$, the compactification of the moduli space of flows is the moduli space of flows, so there are only a finite number of flows between p and q. Secondly, the theorem gives us a very tangible structure on the broken flows between p and q if their index differs by 2. Then we have that the broken flows correspond to exactly the boundary, and are given by

$$\partial\overline{\mathcal{M}(p,q)} = \bigcup_{\mathrm{ind}(r)=\mathrm{ind}(p)-1} \overline{\mathcal{M}(p,r)} \times \overline{\mathcal{M}(r,q)}$$

This fact will become extremely useful later when defining the Morse complex, which is the next thing on our list to do.

0.4 The Morse Complex

The Morse complex is a chain complex whose chain groups are given by critical points of index i, and whose chain maps are going to be defined using properties of the flow lines between critical points. We can already see that if we define the chain group $C_i^M = \mathbb{Z}\ \mathrm{Crit}_i(f)$, then the Euler characteristic of this chain complex should correspond to the Euler characteristic of the manifold. So it looks like we are going the right direction. Let's make this formal.

Definition 0.12 Given a Riemannian manifold (M, g) and a Morse-Smail function f, define the **Morse complex** $C_\bullet^M(M, g, f)$ to be the chain complex with groups

$$C^M(M, g, f) := \mathbb{Z}\,\mathrm{Crit}_i(f)$$

and differential $\partial_i^m : C_i \to C_{i-1}$ to be the signed counting of gradient flow lines. We define it on the generators $p \in \mathrm{Crit}_i(f)$ by

$$\partial_i^M(p) := \sum_{q \in \mathrm{Crit}_{i-1}(f)} \#\mathcal{M}(p,q) \cdot q$$

As notation, we will frequently denote $C_\bullet^M(M, g, f)$ by $C_\bullet^M(f, g)$.

Lemma 0.13 Let M be a manifold, and (f, g) a Morse-Smale pair. Then (as claimed), $C^M_\bullet(f, g)$ is a chain complex.

Proof. We need to show that differential squared is equal to zero. Let $\operatorname{ind}(p) = i$, and $\operatorname{ind}(q) = i - 2$. Then the contribution of the differential from p to q is given by

$$\sum_{r \in \mathrm{Crit}_{i-1}(f)} \#(\mathcal{M}(p, r)) \times \#(\mathcal{M}(r, q) = \#\partial\overline{\mathcal{M}(p, q)}$$

As the signed number of points on a compact 1-manifold is 0, we have that $\partial^2 = 0$.

Example 0.14 We now calculate the Morse homology for the sphere, given the Morse Smale pair of Figure 1. We have that

$$C_2 = \mathbb{Z}\{x_1, x_2\} \qquad C_1 = \mathbb{Z}\{y\} \qquad C_0 = \mathbb{Z}\{z\}$$

The differential is defined as follows. We can define the orientation so that $\partial(x_1) = \partial(x_2) = y$, as there is only one flow line from x_1 or x_2 to y. No matter what orientations we give the descending spaces of y and z, we have that $\partial(y) = 0$, as the two flow lines will inherit opposite orientation. Therefore

$$H^M_2(S^2) = \mathbb{Z} \qquad H^M_1 = 0 \qquad H^M_0 = \mathbb{Z}$$

Of course, the tricky thing to show is that what we have defined here is not dependent on the choice of Morse-Smale pair. There are two different ways that you can show this. One way is to show that this theory is identical to your favorite homology theory. However, it is more useful for this exposition to show that Morse theory can produce results without relying on intuition from other homology theories ,as these techniques will prove useful when looking at Floer theory later.
Let M be a manifold, and (f_0, g_0) and (f_1, g_1) be two Morse-Smale pairs, with associated chain complexes $(C^0_\bullet, \partial_0)$ and $(C^1_\bullet, \partial_1)$. Let $\Gamma = (f_t, g_t)$ be a path of functions and metrics. Our goal is to associate a flow to this path so that we can map critical points of f_0 to critical points of f_1. Start by considering the vector field on $[0, 1] \times M$

$$V := h\frac{\partial}{\partial t} - \nabla^{g_t} f_t$$

where by h we mean the function $(t + 1)^2(t - 1)^2/4$, and $\nabla^{g_t} f_t$ is the gradient of f_t with respect to the metric g_t. Notice that the first part of the function has a critical point of index 1 at $t = 0$ and a critical point of index 0 at $t = 1$ (due to the way that we picked h.) The second part of the function, $\nabla^{g_t} f_t$ achieves critical points at $t = 0$ and $t = 1$ exactly at the critical points of f_0 and f_1 respectively. Therefore, the critical points of this vector field are

$$\mathrm{Crit}_i(V) = \{0 \times \mathrm{Crit}_{i-1}(f_0) \bigcup \{1\} \times \mathrm{Crit}_i(f_1)$$

Let $\mathcal{M}_V((0, p_0), (1, q_1))$ be the set of flow lines from p_0 to q_1 along the flow of V.[1] Define the **Continuation map** $\Phi_\Gamma : C^0_\bullet \to C^1_\bullet$ on generators

$$\Phi_\Gamma(p) := \sum_{q \in \mathrm{Crit}_i(f_1)} = \#\mathcal{M}_V((0, p), (1, q)) \cdot q$$

Lemma 0.15 Φ_Γ is a chain map.

Proof. Given critical points $p \in \mathrm{Crit}_i(f_0)$ and $q \in \mathrm{Crit}_{i-1} \in f_1$, have critical points $(0, p)$ and $(1, q)$ of $[0, 1] \times M$ with respect to Γ. Therefore, $\mathcal{M}_V((0, p), (1, q))$ is a one dimensional manifold (as in $[0, 1] \times M$ p and q correspond to critical points differing in index by 2), and its compactification must have the number

[1]Here I've swept a lot of stuff under the rug. Specifically, we need V to ascending and descending manifolds that intersect transversely. One can show that this is a generic property of Γ, provided that (f_0, g_0) and (f_1, g_1) are themselves Morse-Smale. Also, even if Γ has this nice property, (f_t, g_t) are not necessarily Morse-Smale– in fact, they need to be more interesting for the homotopy Γ to be non-trivial. The times t for which (f_t, g_t) are not Morse Smale are called "bifurcation times", and correspond to a single critical point splitting into two critical points

of endpoints equal to 0 when counted with sign. The boundary of $\overline{\mathcal{M}_V((0,p),(1,q))}$ is determined by critical points that are of index one greater than p. As the critical points occur either when $t = 0$ or $t = 1$, we have that

$$\begin{aligned}\partial\overline{\mathcal{M}_V((0,p),(1,q))} = &\bigcup r \in \mathrm{Crit}_i(f_1)\mathcal{M}_V((0,p),(1,r)) \times \mathcal{M}_{f_1}(r,q)\\ &\cup \bigcup r \in \mathrm{Crit}_{i-1}(f_0)\mathcal{M}_{f_0}(p,r)) \times \mathcal{M}_V((0,r),(1,q))\end{aligned}$$

One can check that this exactly gives you that the difference of $\Phi_\Gamma \nabla_0$ and $\nabla_1 \Phi_\Gamma$ is zero.

Now that we have created a map between chain complexes, we want to show that this map is an isomorphism on homology. We do it by showing that it is chain homotopic to the identity.

Lemma 0.16 Let $\{f_d, g_d | d \in D\}$ be a homotopy between Γ and Γ'. Here D is a digon (closed 2 manifold with two edges and two vertices). Then this induces a chain homotopy between Φ_Γ and Φ'_Γ.

Proof. Give D the metric $\hat{g}$ so that the edges of F have length 1. Let $\hat{f} : D \to \mathbb{R}$ be a function with index 2 critical point at one vertex of D and an index 0 critical point at the other vertex, and no other critical points. Additionally choose $\hat{f}$ such that the negative gradient of $\hat{f}$ with respect to $\hat{g}$ is tangent to the edges and agrees with the negative gradient of $h(t)$ that rises from Γ and Γ'. Then define a vector field V on $D \times M$ by

$$X := \hat{X} + X_d$$

where X_d is the negative gradient of f_d with respect to g_d. Let p_0 be an index i critical point of f_0 and q_1 an index $i+1$ critical point of f_1. Then look at the moduli space of $\mathcal{M}(p_0, q_1)$ of flow lines between p_0 and q_1 in $D \times M$. This should be a zero dimensional space, as the index of p_0 in $D \times M$ is $i+2$, while the index of q_1 in $D \times M$ is $i+1$. This gives us a map from $K : C_\bullet(M, f_0, g_0) \to C_{\bullet+1}(M, f_1, g_1)$. We want to show that this is a chain homotopy between Φ_Γ and $\Phi_{\Gamma'}$.
Let's get a better handle on $\nabla_1 K + K\nabla_0$. Let p_0 and q_1 be critical points of index i in (f_0, g_0) and (f_1, f_1). One can check that this is the signed boundary of the moduli space of flows between f_0 and f_1 in $D \times M$, less the number of broken flows that are broken along the edges of D. The flows broken along the edges of D correspond exactly to the flow points between p_0 and q_1 that can be attributed to either Φ_Γ or $\Phi_{\Gamma'}$, depending on which edge you go along.
Another way to view this is to classify the flows between p_0 and q_1. The flow space should be 1 dimensional in $D \times M$. We know that $\#\{\nabla\overline{\mathcal{M}(p_0, q_0)}\} = 0$. On the other hand, we can correspond each point in the boundary of the compactification with one of 4 types of broken flow lines, corresponding to 4 different types of critical points of index $i+1$ in $D \times M$.

1. Flows that go from p_0 (which is index i in (f_0, g_0)), to a point of index $i-1$ in (f_0, g_0), then to the point q_1. This contribution corresponds to $K\partial_0$.
2. Flows that go from p_0 (which is index i in (f_0, g_0)), to a point of index $i+1$ in (f_1, f_1), then to the point q_1. This contribution corresponds to $K\partial_0$.
3. Flows that go from p_0 (which is index i in (f_0, g_0)) to a point of index $i+1$ along the boundary of D corresponding to the homotopy Γ, and then to the point q_1. This contribution corresponds to Φ_Γ
4. Flows that go from p_0 (which is index i in (f_0, g_0)) to a point of index $i+1$ along the boundary of D corresponding to the homotopy Γ', and then to the point q_1. This contribution corresponds to $\Phi_{\Gamma'}$

This shows us that we have homotopy $\mod 2$, which should be enough for us to believe that the orientations will make everything work so that $\partial K + K\partial = \Phi_\Gamma - \Phi_{\Gamma'}$.

Lemma 0.17 $\Phi_{\Gamma_2 \times \Gamma_1}$ is chain homotopic to $\Phi_{\Gamma_2} \circ \Phi_{\Gamma_1}$

Proof. Apparently the above proof works, but you replace "digon" everywhere with triangle" or something.

Theorem 0.18 $H^m(M, f, g)$ is independent of choice of f or g.

Proof. Let Φ be a homotopy from f_0 to f_1, and Φ' a homotopy from f_1 to f_0. Then $\Phi' \circ \Phi$ is a homotopy from f_0 to f_0, and may be assumed to be the identity by Lemma 0.4. As a result, $\mathrm{id} = \Gamma_{\Phi' \circ \Phi} = \Gamma_{\Phi'} \circ \Gamma_{\Phi}$ on homology. Similarly, $\mathrm{id} = \Gamma_{\Phi \circ \Phi'} = \Gamma_{\Phi} \circ \Gamma_{\Phi'}$ on homology. So Γ_{Φ} is an isomorphism on homology. So we have no dependence on choice of f_0 and f_1.

References

[1] Michael Hutchings. Lecture notes on morse homology (with an eye towards floer theory and pseudo-holomorphic curves). Lecture Notes, 2002.