Categorification

Jeff Hicks

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1 Plan of Attack

Categorification is a technique used in algebraic combinatorics to extend classical combinatorial objects that we might be interested to contain more data. Categorification is not some set process; but there are some general themes. Here are two kind of different approaches to the problem:

Enumerative Combinatorics	**Category Theory**
A number that counts things!	Look, an object in some category
An equation of numbers!	An isomorphism of objects!
A function between two objects	Functors!

Why would we want to do this? Because categories are a lot more expressive than numbers are, and contain a lot more data. While this is not a formal definition, here is a rough idea of what we will do.

Definition 1.1 Suppose we have some problem in combinatorics. Then we can **categorify it** by creating an associated problem using objects and morphism. We also have a method to **decategorify** our categorification to obtain the original combinatorial data.

Maybe a good diagram to think about is something like this:

$$\begin{tikzcd} \text{Categoy} \arrow[r, "categorification"] \arrow[d, "f"] & \text{Improved Category} \arrow[d, "\mathcal{F}"] \\ \text{Invariants} & \text{Category of Invariants} \arrow[l, "decategorification"'] \end{tikzcd}$$

While categorification is more of a philosophy than an exact method, we will look at a type of categorification that shows up often in mathematics, which is the "natural upgrade" to the idea of inclusion/exclusion.
Step 1 The first thing we'll need is some set of objects, and an invariant of those objects we want to study. For example, in enumerative combinatorics we might be looking at something as simple as "the size of a set". Usually we will be interested in a function that takes values in some ring.
Step 2 Usually the next upgrade is to obtain some kind of decomposition rule that allows us to compute that invariant in terms of simpler objects, for example

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Step 3 By iterating this process over and over we can obtain a rule that is more powerful, like the principle of inclusion/ exclusion.

Theorem 1.2 Inclusion-Exclusion

Let $\{A_i\}_{i\in I}$ be a collection of sets. Then the principle of inclusion/ exclusion states that

$$\left|\bigcup_{i\in I} A_i\right| = \sum_{\alpha\subset I} (-1)^{|\alpha|+1} \left|\bigcap_{i\in\alpha} A_i\right|$$

Step 4 The next step is categorification, which we are going to get into next! Categorification is going to tell us exactly how exactly the sets in inclusion/exclusion relate to each other. Right now, with our example with sets, we only no two parts of this pieces.

$$\begin{tikzcd} \textbf{Set} \arrow[r, "?"] \arrow[d, "|\cdot|"] & \text{Improved Category ?} \arrow[d, "\mathcal{F}"] \\ \mathbb{N} & \text{Category of Invariants ?} \arrow[l, "?"] \end{tikzcd}$$

Why do we like categorification?

1. Categorification gives us a functor, rather than function as an invariant.
2. When our decomposition rules don't work, categorification tells us exactly how our decomposition rule fails.
3. Categorification is going to give us many invariants from one. Figuring out what these new invariants mean can give us new directions to search in a field. Frequently we can construct new categorical invariants without knowing where they come

2 Categorification

2.1 Building Better Categories

Suppose I have some category $\mathcal{C}$. If you want an example to keep in mind, think of the category of sets. Our goal here is to create a better category where talking about things like chain complexes make sense. So we are going to upgrade our category to have more and more of the structures that we are interested in studying. Let's start by changing the morphisms of our category.

Definition 2.1

Let $\mathcal{C}$ be a category. Define **enrichment of $\mathcal{C}$ over $\mathbb{Z}$** to be the category $\mathcal{C}^+$ where the objects are unchanged;

$$\textbf{Ob}(\mathcal{C}^+) = \textbf{Ob}(\mathcal{C})$$

but the morphisms have been extended by $\mathbb{Z}$ linearity. That is, for all $A, B \in \textbf{Ob}\mathcal{C}^+$, we have

$$\hom_{\mathcal{C}^+}(A,B) = \bigoplus_{f\in\hom_{\mathcal{C}}(A,B)} \mathbb{Z}_f$$

and we define the composition rule by extending linearity

$$(f_1 + f_2) \circ (g_1 + g_2) = (f_1 \circ g_1 + f_1 \circ g_2 + f_2 \circ g_1 + f_2 \circ g_2).$$

For those who have seen a little category theory, you'll notice that $\mathcal{C}^+$ is a preadditive category. Of course, it doesn't have all the properties we want yet— for instance, there is no zero object, no ability to take direct sums of objects, no ideas of kernels. We want a category we have all of these operations.

Definition 2.2 Given a preadditive category $\mathbb{C}^+$, define the **additive closure of Mat**$(\mathcal{C}^+)$ to be the category where

- the objects are formal direct sums of objects in $\mathcal{C}^+$
- If $A = \bigoplus_{i=1}^n A_i$ and $B = \bigoplus_{j=1}^n B_j$ are two objects of the additive closure, then a morphism between them is a $m \times n$ matrix where the i, j entry is a morphism between A_i and B_j.
- Composition of morphisms is done via matrix multiplication.

Now we know what to put in the top right corner of our categorification

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "categorification"] \arrow[d, "f"] & \text{Mat}\mathcal{C}^+ \arrow[d, "\mathcal{F}"] \\
\text{Invariants} & \text{Category of Invariants} \arrow[l, "decategorification"]
\end{tikzcd}$$

Example 2.3 If you do this process starting with $\mathcal{C}$ = **Set**, what we will get is equivalent is the category of free abelian groups. When we try categorifying, we'll get

$$\begin{tikzcd}
\textbf{Set} \arrow[r, "categorification"] \arrow[d, "f"] & \text{Free Abelian Groups} \arrow[d, "i"] \\
\mathbb{N} & \text{Category of Invariants} \arrow[l, "decategorification"]
\end{tikzcd}$$

Now lets fill in the other parts of the diagram.

$$\begin{tikzcd}
\textbf{Set} \arrow[r, "categorification"] \arrow[d, "f"] & \text{Free Abelian Groups} \arrow[d, "i"] \\
\mathbb{N} & \text{AbGrps} \arrow[l, "rank"]
\end{tikzcd}$$

You can check that this works out well, and that the diagram commutes and everything.

Why would we want to do this process at all? It seems like we haven't done much yet. The nice thing we have is that instead of having a function f that defines an invariant, we have a functor. And we have all kinds of techniques to study functors.

2.2 Resolutions

When we studied enumerative combinatorics, one our techniques was to look at decompositions of our objects that we were studying. For instance, with sets we might take a set $A = \cup_i A_i$ as a decomposition.

What should the proper thing with categories? We should take a resolution.

$$
\begin{tikzcd}
A_1 \arrow[drr] & A_2 \arrow[dr] & \dots & A_{k-1} \arrow[dl] & A_k \arrow[dll] \\
 & & A & &
\end{tikzcd}
$$

Of course, what we should look at is not just this first step of the resolution. From last talk, we know that things that are really worth looking at are projective resolutions. It's a little weird to ask what a "projective resolution" should be, but lets run through an example to see what we mean.

Example 2.4 Let's go back to the case of sets. From here on out, I am working with the "categorification" of the set. Let $A = A_1 \cup A_2 \cup A_3$. Then a resolution of A looks like this

$$
\begin{tikzcd}
 & A_1 \cap A_2 \arrow[r] \arrow[ddr] & A_1 \arrow[ddr] & \\
 & \oplus & \oplus & \\
A_1 \cap A_2 \cap A_3 \arrow[r] \arrow[uur] \arrow[ddr] & A_1 \cap A_3 \arrow[uur] \arrow[ddr] & A_2 \arrow[r] & A_1 \cup A_2 \cup A_3 \\
 & \oplus & \oplus & \\
 & A_2 \cap A_3 \arrow[r] \arrow[uur] & A_3 \arrow[uur] &
\end{tikzcd}
$$

When we are given a resolution and a functor $\mathcal{F}$ and a resolution, we should hope that the functor $\mathcal{F}$ is right exact, and apply the functor to the complex to get a chain complex.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "Cat"] \arrow[d, "f"] & \text{Mat}\mathcal{C}^+ \arrow[r, "Res^\bullet"] \arrow[d, "\mathcal{F}"] & Ch(\text{Mat}\mathcal{C}^+) \arrow[d, "\mathcal{F}"] \\
\text{Invariants} & \text{Category of Invariants} \arrow[l, "DeCat"] & Ch(\text{ Invariants})
\end{tikzcd}
$$

Now how do resolutions relate back to the problem at hand? We have two things that we can do to this complex. The first thing that we could do is compute it's Euler characteristic, which would give us data on how our object fails to be projective.

Definition 2.5 Suppose we have an object $X \in \mathcal{C}$, and let $Res^\bullet(Cat(X))$ be its projective resolution. Then define the **Euler Characteristic** to be

$$\chi(X) = \sum_i (-1)^i DeCat(Res^i(Cat(X))$$

and the **Reduced Euler Characteristic** to be

$$\bar{\chi}(X) = f(X) - \chi(x)$$

Now what do these things mean? The reduced Euler Characteristic should measure how far our object is from being projective. Notice that the Euler characteristic and the reduced Euler characteristic do not use the differential map on the chain complex in their definition, so we aren't really capturing the full power of categorification. What we should really do is apply homology to our chain complex, and get a large number of invariants.

So the amount of data that we can extract from an invariant has been extended. Going back to our model of information that categorization produces, have this set up (this is **not** a commutative diagram, it's just showing the flow of information!)

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "Cat"] \arrow[d, "f"] & \mathrm{Mat}\mathcal{C}^+ \arrow[r, "Res^\bullet"] \arrow[d, "\mathcal{F}"] & Ch(\mathrm{Mat}\mathcal{C}^+) \arrow[d, "\mathcal{F}"] \\
\text{Invariants} & \text{Category of Invariants} \arrow[l, "Decat"] & Ch(\text{ Invariants}) \arrow[ll, bend left, "\chi{,}\bar{\chi}"] \arrow[d, "H^i"] \\
\text{New indexed Invariants} & & \text{Category of Invariants} \arrow[ll, "Decat"]
\end{tikzcd}$$

Now there are some problems, as we now have information stored in a chain complex. Ideally, this chain complex contains some additional information about the object we are studying. To remove that information in a meaningful fashion, we can take homology and decategorify. [1] Let's say if f is the function we are interested in, then f^i is the new indexed invariant. This looks a lot like the constructions we had with derived functors. However, instead of looking at the failure of the functor $\mathcal{F}$ from being exact, this measures some kind of failure of projective objects in this additive category having the same properties of projectives in abelian categories.

One last question we might have is how our newly indexed invariants relate to our old invariants. In the case of chain complexes and dimension, we will get a fundamental lemma.

Lemma 2.6 (Fundamental Lemma) Let $C_\bullet$ be a chain complex. Then

$$\sum_i (-1)^i \dim(C_i) = \sum_i (-1)^i \dim(H_i(C_\bullet))$$

We therefore expect that the alternating sum of our newly indexed invariants to be the Euler characteristic.

2.3 Examples

Perhaps the best way to see the power of this techniques is with a few examples.

Example 2.7 We've already seen that the function $|\cdot| : \mathbf{Set} \to \mathbb{N}$ that for every decomposition $\bigcup_i A_i = A$ we get at a projective resolution of A.

$$\begin{tikzcd}
 & A_1 \cap A_2 \arrow[r] \arrow[ddr] & A_1 \arrow[ddr] & \\
 & \oplus & \oplus & \\
A_1 \cap A_2 \cap A_3 \arrow[r] \arrow[uur] \arrow[ddr] & A_1 \cap A_3 \arrow[uur] \arrow[ddr] & A_2 \arrow[r] & A_1 \cup A_2 \cup A_3 \\
 & \oplus & \oplus & \\
 & A_2 \cap A_3 \arrow[r] \arrow[uur] & A_3 \arrow[uur] &
\end{tikzcd}$$

[1] Even if we cannot decategorify, we would like something like the choice of projective resolution does not change the homotopy type of the complex constructed

Applying the functor that assigns to each set the free group generated on its elements

$$\begin{tikzcd}
 & \mathbb{Z}\{A_1 \cap A_2\} \arrow[r] \arrow[rdd] & \mathbb{Z}\{A_1\} \arrow[rdd] & \\
 & \oplus & \oplus & \\
\mathbb{Z}\{A_1 \cap A_2 \cap A_3\} \arrow[ruu] \arrow[r] \arrow[rdd] & \mathbb{Z}\{A_1 \cap A_3\} \arrow[ruu] \arrow[rdd] & \mathbb{Z}\{A_2\} \arrow[r] & \mathbb{Z}\{A_1 \cup A_2 \cup A_3\} \\
 & \oplus & \oplus & \\
 & \mathbb{Z}\{A_2 \cap A_3\} \arrow[r] \arrow[ruu] & \mathbb{Z}\{A_3\} \arrow[ruu] &
\end{tikzcd}$$

This is an exact sequence, so the homology is zero everywhere. We can now prove the principle of Exclusion/ Inclusion. From the fundamental lemma,

$$0 = \sum_i (-1)^i \dim(H^i(Ch(A)) = \sum_i (-1)^i \dim(Ch_i(A)) = \sum_\alpha (-1)^{|\alpha|} |\bigcap_{i \in \alpha} A_i|.$$

Intuitively, this should tell us that "set projectives" have exactly the same property as "abelian group projectives". Notice that to the decomposition rule

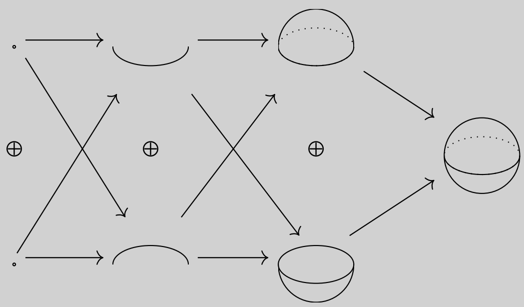
$$|A \cup B| = |A| + |B| - |A \cap B|$$

We get an associated exact sequence of invariants

$$0 \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(A \cup B) \to 0$$

There are other examples.

Example 2.8 Given a space, have a function that counts the number of connected components, $H^0(X)$. A projective object in this category is a contractible space. So take a space, cover it with contractible spaces, and apply this homology trick. Let's actually write one of these out, and compute the homology of S^n.



When we apply the H^0 functor to this, we get the following diagram

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[rdd] & \mathbb{Z} \arrow[r] \arrow[rdd] & \mathbb{Z} \arrow[rd] & \\
\oplus & \oplus & \oplus & \mathbb{Z} \\
\mathbb{Z} \arrow[r] \arrow[ruu] & \mathbb{Z} \arrow[r] \arrow[ruu] & \mathbb{Z} \arrow[ru] &
\end{tikzcd}$$

Taking homology gives us

$$\mathbb{Z} \qquad 0 \qquad 0 \qquad 0$$

Depending whether you choose to keep to $\mathbb{Z}$ on the right, or toss it out, you get either cohomology or reduced cohomology of S^n. Notice that $H^0(X)$ satisfies the decomposition relation

$$H^0(A \cup B) = H^0(A) + H^0(B) - H^0(A \cap B) + \text{Some Error Term related to the number of holes in } A \cup B$$

and we have an associated long exact sequence on homology which tells us where this error term is coming from:

$$0 \to H^0(A \cup B) \to H^0(A) \oplus H^0(B) \to H^0(A \cap B) \to H^1(A \cup B) \to \dots$$

Before we categorified H^0, we knew how to compute it in terms of decompositions, but it was a bit difficult to say why the principle of inclusion/exclusion failed for homology. Now we know why it fails, and exactly where it fails. Again, we expect the reduced cohomology of contractible spaces to be 0 because these spaces are projective.

Notice that it is not immediately obvious what these homology invariants tell us; we know that they exist, and they give us additional data, but it is not immediately clear what data these invariants capture. Understanding what these invariants mean is a deep problem in the fields that use them.
So far we have motivated this with examples where the resolutions are obvious to look at , but now we are going to attack a slightly harder problem.

2.4 Takeaways

What have we done here? We've managed to create a number new pieces of data from old ones.

1. We've taken our invariants and turned it into something functorial that respects functions.
2. We've created a measure of how far from "projective" an object is, (whatever that means)
3. We've gotten a bunch of new measurements associated to an object, but it is immediately clear what they mean. In this way, we've created a bunch of new questions for us to study.

3 Khovanov Homology

3.1 Background on Knots

Our initial goal for this section is to develop a function

$$\text{Knots} \xrightarrow{J} \mathbb{Z}(x)$$

called the Kauffman polynomial. Before we can get into this, we should have some basic definitions. While the definition of what a knot should be varies between texts, this is the one that we are going to be using:

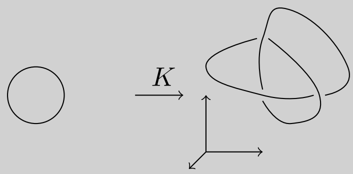
Definition 3.1

A **knot** is a smooth embedding of the circle $K : S^1 \to \mathbb{R}^3$ which extends to an embedding of the torus $S^1 \times D^2 \to \mathbb{R}^3$.[2]Two knots are defined be **equivalent** if there exists a smooth isotopy between them. Define **Knot** to be the set of equivalence classes of knots. A projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ produces a knot diagram of K is $(\pi \circ K)$ is an immersion and $|(\pi \circ K)^{-1}(x)| \leq 2$. We say a **knot diagram** of K is the subset $Im(\pi \circ K)$ along with crossing data for each x such that $(\pi \circ K)^{-1}(x) = 2$.

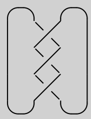
While this sounds really difficult, it's not so bad to look at.

[2]Some people call this a tame knot

Example 3.2 Here we are going to describe what a trefoil knot is, and what its knot diagram is. The trefoil knot is the actual embedding of a circle into three dimensional space.

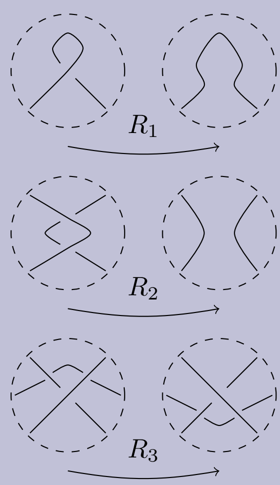


A knot diagram is a representation of that picture in the plane.



We usually work with knot diagrams instead of knots, because they are so much easier to talk about– we can represent them with two dimensional pictures. For convenience, from now on we will not distinguish between a knot and its diagram To get between different knot diagrams, we can apply transformations of a knot diagram.

Definition 3.3 Let K be a knot diagram. Then on a small section of the knot diagram, we can apply one of the following three changes to the diagram:



These are called the **Reidimeister Moves**.

We can change one knot diagram into another by applying the Reidimeister moves to the diagram. It is clear to see that applying a Reidimeister move corresponds to a smooth isotopy of knots.

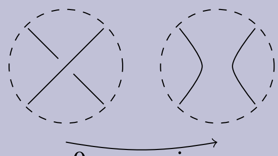
Lemma 3.4 If K_1 and K_2 represent two equivalent knots, then there is a sequence of Reidimeister moves than changes K_1 to K_2.

One common strategy for creating a function from f : **Knots** $\to \mathcal{C}$ is to create a function on knot diagrams, and show that $f(K)$ remains unchanged under applications of the Reidimeister moves. We will do this to define the Kauffman polynomial.

3.2 The Kauffman Polynomial

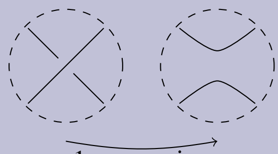
The Kauffman Polynomial is a polynomial that we can assign to a knot diagram. Before we can define the Kauffman polynomial, we are going to have to define a way to simplify knots.

Definition 3.5 Let K be a knot diagram with an assigned orientation. Let x be a crossing in the knot diagram. We define the 0-uncrossing of K at x to be the diagram where we have replaced the crossing x in the following fashion:



0-uncrossing

and the 1-uncrossing of K at x to be the diagram where we replaced the crossing x in the following fashion.

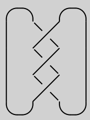
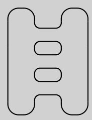
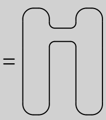


1-uncrossing

Given a knot diagram K with an ordering of crossings, define $K_{x_1 x_2 \ldots x_n}$ be the diagram where the ith crossing is x_i-uncrossed.

This is probably best seen via an example.

Example 3.6 Let K be the trefoil. Then here are a few uncrossing of the trefoil.

K_{xxx} =  K_{000} =  K_{011} = 

We define the Kauffman polynomial via induction.

Definition 3.7 Let K be a knot diagram, and let K_0 and K_1 be the 0 and 1 uncrossing of K at some selected crossing x of K. Then define

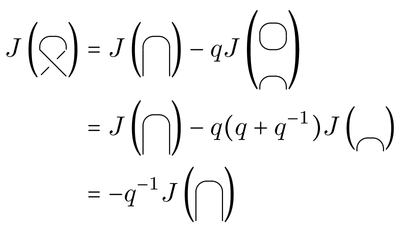
$$J(K) = J(K_0) - qJ(K_1)$$

and if K is a diagram of n unknots, define $J(K) = (q + q^{-1})^n$.

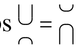
Claim 3.8 $J(K)$ is an invariant up to a multiple of q.

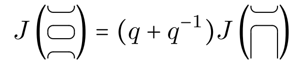
Proof. The idea is to show that the three Reidimeister moves only change the Jones polynomial of a link by a degree of q or by a sign change. We will show the invariance for each of the R-moves.

1. We show invariance under the first R-move.

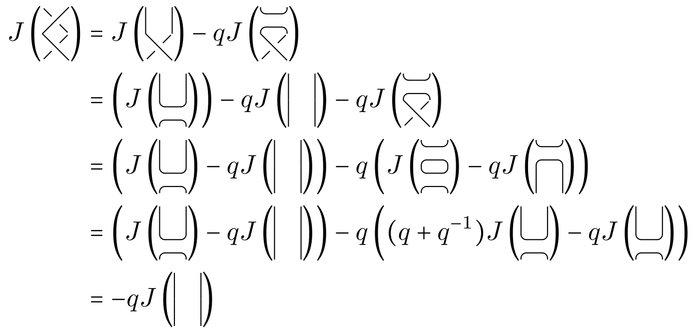


2. We show invariance under the second R-move.

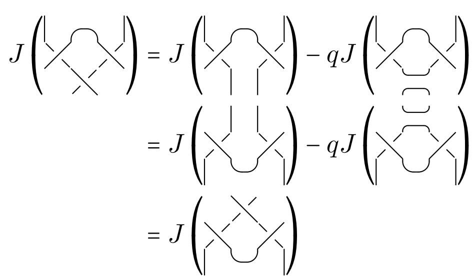
With the relationships  and



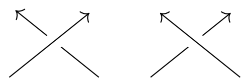
we perform the computation of the second Reidimeister move:



3. We show invariance under the third R-move. This one follows from the second Reidimeister move.



We now have shown that the Jone's polynomial is an invariant of the knot up to a sign and a multiple of q. There is a way to correct this shifting. If we assign an orientation to the knot, we can now label crossings in two different ways, as in the below figure

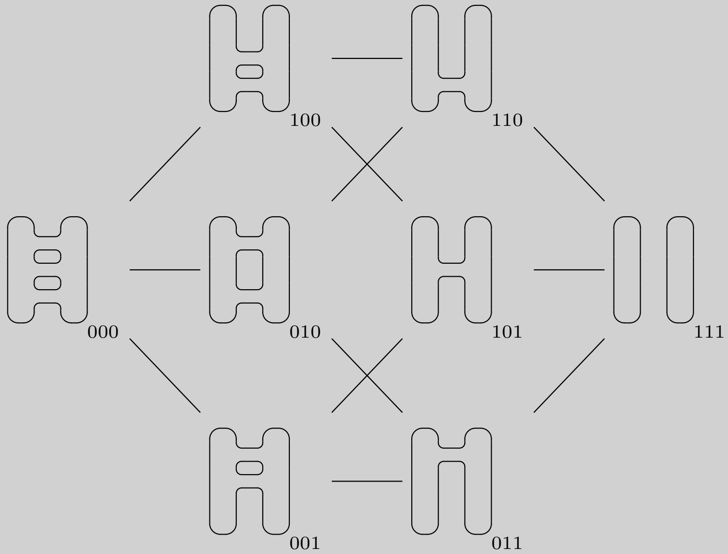


Let n_+ and n_- count the number of + and − crossings respectively.We see that if we multiply our J polynomial by a factor of $(-1)^{n_-}q^{n_+-2n_-}$ that this new polynomial now a complete invariant of the knot.
If we try and compute the Jone's polynomial of a knot by continually taking 0 and 1 resolutions, we see that we reach every possible smoothing of the knot. This means that if our original knot diagram has n crossings, there are 2^n different resolutions of the knot. We see that for each 1 resolution the knot contains, it picks up factor of $-q$. This gives us a simpler way to compute the jones polynomial. The Jones polynomial is given by

$$\sum_{i\in\{0,1\}^n}(-q)^{|i|}J(K_i)$$

where K_i means the resolution where the crossings are given by the string i, and $|i|$ counts the number of times 1 appears in the string i. Let's do a sample computation.

Example 3.9 Let's do a decomposition of a the trefoil into all of its smaller components.



Each vertex on the cube is a smoothing of the knot, and the smoothings are ordered from left to right in columns corresponding to the number of 0 smoothings chosen. We draw an arrow between two vertices if they differ by just a single place in their smoothing. We now create a cube that lists each resolution's Jone's polynomial times $(-q)^i$, where i is the column the polynomial lies in.

$$\begin{tikzcd}
 & q(q+q^{-1})^2 \arrow[r] \arrow[dr] & q^2(q+q^{-1})^1 \arrow[dr] & \\
(q+q^{-1})^3 \arrow[ur] \arrow[r] \arrow[dr] & q(q+q^{-1})^2 \arrow[ur] \arrow[dr] & q^2(q+q^{-1})^1 \arrow[r] & q^3(q+q^{-1})^2 \\
 & q(q+q^{-1})^2 \arrow[r] \arrow[ur] & q^2(q+q^{-1})^1 \arrow[ur] &
\end{tikzcd}$$

We now can compute the unnormalized Jones polynomial: it is just the alternating sum of the columns of this cube. That is, the unnormalized Jones polynomial of the trefoil is $(q+q^{-1})^3 - 3q(q+q^{-1})^2 + 3q^2(q+q^{-1})^3 - q^3(q+q^{-1})^2 = q^{-2} + 1 + q^2 - q^6$

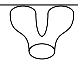
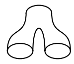
We now have a classical invariant that we can assign to a polynomial. This is a bit of a strange invariant, and it is not immediately obvious why this should be the right invariant to look at. Its construction seems a little arbitrary, and we aren't really sure what kind of data it encapsulates. It is now known that the Jones polynomial comes out of Chern Simmons Theory.
Now that we have the Jones polynomial, let's go and categorify it.

3.3 Overview of categorification

In order to categorify the Jones polynomial, we are going to have to come up with a category that knots sit in. The category that we are going to look at are link cobordisms.

Definition 3.10 Let $L_1, L_2 \subset R^3$ be links. Then a **Link Cobordism** between L_1 and L_2 is an embedded surface $\Sigma \subset \mathbb{R}^3 \times I$ such that $\partial\Sigma = L_1 \sqcup L_2$. The category of links with morphism given by Link cobordisms will be **Cob** which are defined up to cobordism equivalence.

Notice that the identity cobordism in this category is just a cylinder.
Cobordisms have the nice property that they can be decomposed into elementary cobordisms, which are called pants, copants, unit and counit.

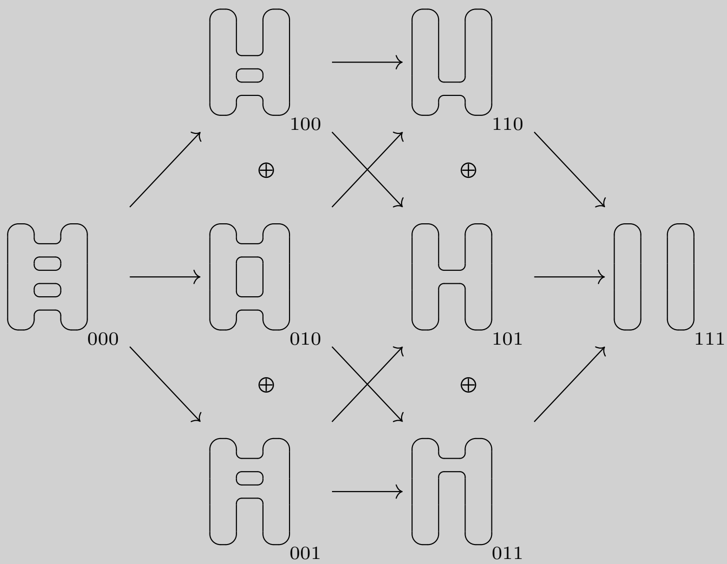
Principle	Name	2D Cobordism	Algebraic Operation
Merging	Copants		$m : V \otimes V \to V$
Creation	Unit		$i : R \to V$
Splitting	Pants		$\Delta : V \to V \otimes V$
Annihilation	Counit		$\epsilon : V \to R$

The idea that we can break down cobordism into their elementary components will be very important later. Let's return to the task of categorification. Our categorification should be based on resolutions in this category. Fortunately, we already have idea of what resolutions should be, as we've already constructed one to compute the Kauffmann bracket.
For convenience, from now on we will denote the category Mat**Cob**$^+ = \mathcal{C}$ from now on. What we are interested in doing is producing a "projective resolution" for a knot in $\mathcal{C}$. We define the resolution of a knot diagram K to be the chain complex in $Ch\mathcal{C}$ with the following objects.
Pick an ordering of the crossings of K. Suppose that K has n crossings. For each $\alpha \in \{0,1\}^n$, define the α-resolution K_α to be the diagram where the i crossing of K is given the uncrossing specified by the i-th component of α. Suppose that $\alpha < \alpha'$ in the poset $\{0,1\}^n$. Then define the map $d_{\alpha\alpha'} : K_\alpha \to K_{\alpha'}$ as follows. Suppose $\alpha < \alpha'$ differ at the ith place. Then define the cobordism $d_{\alpha\alpha'}$ to be identity cobordism on all of the components of K_α that do not "touch" the ith crossing, and to be the "pants" cobordism on the components that are different between the two resolutions.
Define the chain group $Kh^i(K) = \bigoplus_{|\alpha|=i} K_\alpha$, and define the map $d : Kh^i(K) \to Kh^{i+1}(K)$ to be given by

$$d := \bigoplus_{\alpha<\alpha', |\alpha|=i} (-1)^{\sigma(\alpha)} d_{\alpha\alpha'}$$

Then $Kh^\bullet(K)$ is a chain complex in $Ch(\mathcal{C})$. Perhaps the easiest way to see this is to actually look at one of these chain complexes.

Example 3.11 Let's try and compute the chain complex of a trefoil.



$$Kh^0(K) \longrightarrow Kh^1(K) \longrightarrow Kh^2(K) \longrightarrow Kh^3(K)$$

You can check that this is a well defined complex.

Theorem 3.12 Bar-Natan

The homotopy type of this chain complex is independent of diagram chosen (up to a few relations)

We also, as expected, get an exact sequence of some kind related to the decomposition rule, just like we did before for sets and singular homology.

$$0 \to Kh^\bullet(K_0) \to Kh^\bullet(K) \to Kh^\bullet(K_1) \to 0$$

This means we have a well defined map

$$\begin{tikzcd} \mathbf{Cob} \arrow[r] \arrow[d, "J"] & \mathcal{K}om(Mat\mathbf{Cob}^+) \arrow[dl, "?"] \\ \mathbb{Z}[x] & \end{tikzcd}$$

Of course this gives us the categorified invariant. But what is the process of decategorification?

3.4 Graded Modules

Our eventual goal is to construct a homology theory where the Euler characteristic of the chain complex is the chromatic polynomial. Before we construct to complex, we will first review some basic properties of graded dimension.

Definition 3.13 Let $M = \bigoplus_i M_i$ be a decomposition of a graded module M into its homogeneous submodules. Then the **graded dimension** (or sometimes **quantum dimension**) of M is the formal power series

$$\text{qdim} M = \sum_i q^i \text{rk}(M_i)$$

where $\text{rk}(M_i) = \dim(M_i \otimes_R \text{Frac}(R))$

Graded dimension plays very nicely with the operations of tensor product and direct sum. Let A and B be two graded modules. Then we have the two identities

$$\text{qdim}(A \oplus B) = \text{qdim}(A) + \text{qdim}(B)$$

$$\text{qdim}(A \otimes B) = \text{qdim}(A) \cdot \text{qdim}(B)$$

where $A \oplus B$ and $A \otimes B$ are given the natural gradings inherited from A and B.

Example 3.14 We give several examples of quantum grading that will be useful to remember in the future.

1. Let R be a ring. Then the ring $R[x]$ with the conventional grading has $\text{qdim} R = 1 + q + q^2 + \dots$.
2. Let $R[x]$ be as before. We define the module

$$V := R[x]/(x^2)$$

   Then $\text{qdim}(V) = 1 + q$. We will use this module over and over again.
3. Let V be as in the before example. Then

$$\text{qdim}(V^{\otimes n}) = (1 + q)^n$$

We are slowly building up a framework that allows us to do addition and multiplication of polynomials with algebraic spaces, instead of working with a ring. We finally will introduce one more idea.

Definition 3.15

A **graded chain complex** is a chain complex $C^\bullet$ where each C^i is a graded R module and the differentials d^i are graded maps. The homology groups $H^i = \frac{\ker d}{\operatorname{Im} d}$ inherit a grading from the chain complex and we define the **graded Euler characteristic** $\chi(C^\bullet)$ is the alternating sum of the quantum dimensions of the homology $C^\bullet$, that is

$$\chi(C^\bullet) = \sum_i (-1)^i \text{qdim} H^i(C)$$

This definition is not entirely accurate— I've actually put up the definition of a graded cohomology theory. However, in the literature, it is almost always called a graded homology theory, and you could change the definition to be "proper" by inserting a few minus signs into the definition.

3.5 TQFTs

The way to reverse this process is by applying a functor called a TQFT.[3] A TQFT transforms the difficult problem of telling two cobordisms apart to the simpler problem of telling two algebraic objects apart.

Definition 3.16

A **TQFT** is a functor from the category of cobordisms to the category of R-algebras.

Let's go and actually construct one of these functors.

Definition 3.17

We define the following multiplication, co-multiplication, unit and co-unit structure on the R-module V.

- The **multiplication** map, $m : V \otimes V \to V$ is defined as

$$m(1 \otimes 1) = 1 \qquad m(1 \otimes x) = x$$
$$m(x \otimes 1) = x \qquad m(x \otimes x) = 0$$

- The **comultiplication** map, $\Delta : V \to V \otimes V$ is defined as

$$\Delta(1) = 1 \otimes 1 \qquad \Delta(x) = 1 \otimes x + x \otimes 1$$

- The **unit** is the map $\epsilon : R \to V$ by $1 \mapsto 1$
- The **co-unit** is the map $i : V \to R$ by $1 \mapsto 0, x \mapsto 1$.

These maps give V the structure of a **Frobenius algebra**.

Let V be a Frobenius algebra over R. Define the functor $\mathcal{F} : \textbf{Cob} \to R\text{alg}$ as follows. To every diagram D $\mathcal{F}(D) = \otimes_{x \in K(D)} V_x$, where $K(D)$ is the number of connected components of D. In order to define the values of $\mathcal{F}$ on cobordism, we need only know where $\mathcal{F}$ takes elementary cobordisms, and check that it respect the relationship of cobordism equivalence.
Define the values of $\mathcal{F}$ on elementary cobordisms as follows

- If $m_{xx'} : D \to D'$ is an elementary cobordism which merges two connected x and x' together into w, then let $\mathcal{F}(m) : \mathcal{F}(D) \to \mathcal{F}(D')$ be that takes $V_x \otimes V_{x'} \to V_w$ by the multiplication map m of Frobenius algebras. Otherwise, let $\mathcal{F}(m) : \mathcal{F}(D) \to \mathcal{F}(D')$ act by the identity.
- If $\Delta_w : D \to D'$ is an elementary cobordism which splits one connected component into two, then let $\mathcal{F}(\Delta) : \mathcal{F}(D) \to \mathcal{F}(D')$ splitting w into v and v' be that which takes $V_w \to V_v \otimes V_{v'}$.
- If $i : D \to D'$ is the inclusion of an additional component into D', then define $\mathcal{F}(i) : \mathcal{F}(D) \to \mathcal{F}(D')$ to be the unit map of Frobenius algebras.
- If $\epsilon : D \to D'$ is the removal of a connected component, let $\mathcal{F}(\epsilon) : \mathcal{F}(D) \to \mathcal{F}(D')$ be the counit map or Frobenius algebras.

[3]There is a reason why this thing is called this, but I don't know enough physics to tell you why.

Since every cobordism can be decomposed into a series of elementary cobordism, we can define the value of $\mathcal{F}$ on every cobordism by decomposing it into its elementary parts. To show that this map is well defined, we need it to be independent of elementary decomposition.

Theorem 3.18 $\mathcal{F}(B)$ is independent of elementary decomposition chosen for (B).

Proof. It suffices to check that $\mathcal{F}$ respects the elementary cobordism relationships, as we showed that these relationships generated all of the cobordism equivalences. The suggestive names show that required relationships hold— i.e. associativity of cobordism multiplication corresponds to associativity of multiplication in Frobenius algebras.

Corollary 3.19 $\mathcal{F}$: **Cob** → Ralg is a functor.

We now have the functor required for decategorification.

3.6 Decategorification

We have the ability to decategorify now.

Claim 3.20 We have the following commutative diagram

$$\begin{tikzcd} \mathbf{Cob} \arrow[r] \arrow[d, "J"] & Kom(Mat\mathbf{Cob}^{+}) \arrow[d, "TQFT"] \\ \mathbb{Z}(q) & Ch(R-\text{mod}) \arrow[l, "\chi"'] \end{tikzcd}$$

Let's compute an example.

This of course is a little strange. We have a categorification, and an associated Euler characteristic. However, we don't know what the know what the original invariant was! This is a little bit of a mystery, and the meaning of $Kh^0(K)$ is rather deep. However this is what we do know.

1. Khovanov homology descend from a spectral sequence related to Instaton Floer Homology.
2. $Kh^0(K)$ contains data related the Rasmussen s-invariant, which gives us a bound how close the link is to being slice.
3. The bottom and top homology groups of most knot homology theories coming from Floer theory are give us data on the genus of a knot. They also contain data on the fiberedness of the knot.
4. Khovanov homology has recently been constructed as the singular homology of the suspension spectrum of the knot.
5. Khovanov homology can detect the unknot, which is a most remarkable result! It is still not known if the Jones polynomial can. This tells us that the unknot is the only "projective" type of object in the category of cobordisms.

4 Closing Thoughts

Categorification can be applied to many different invariants. Polynomials arising from skein relations seem especially susceptible to the techniques of categorification. Here are a few of the classical invariants that can be upgraded via categorification techniques

Classical Result	**Categorification**
Kauffman Bracket (Knot Theory)	Khovanov Homology
Chromatic Polynomials (Graph Theory& Matroids)	Chromatic Complex
Tutte Polynomials	A variety of methods
Connected Components	Singular Homology
Weyl Character formula	BernsteinGelfandGelfand
Functors	Projective Resolutions
Binomial Coefficients	Exterior Algebras

If you see an inclusion/exclusion type of relation in the future, I hope that this paper gives you some of the required background to try and categorify the object you are studying. If you have more interest in Khovanov homology, or other papers on categorification, I've included some papers below.