MODULE AND FILTERED KNOT HOMOLOGY THEORIES

JEFF HICKS

Abstract. We provide a new way to define Bar-Natan's $\mathbb{F}_2[u]$ knot homology theory. The u torsion of $BN^{\bullet,\bullet}$ is shown to explicitly give Turner's spectral sequence computing the filtered $\mathbb{F}_2$ homology. Finally, we extended Schumakovitch's map in a categorical way to exhibit some further structure on the $\mathbb{F}_2$ and reduced $\mathbb{F}_2$ homology.

Contents

1. Introduction

1.1. **History and Current Literature.** Since Khovanov [Kho99] introduced his homology theory for links in 1999, there has been a lot of progress in categorification of knot polynomials, and investigation on knot homology theories in general. In 2004, Bar-Natan published [Bar04] a description of the Khovanov Bracket, $[[L]]^\bullet$ as a homotopy category over the cobordisms. Thie gave an explicit way to produce new homology theories for knots, via the application of TQFT's to the Khovanov bracket.

In his paper, Bar-Natan made many conjectures on Khovanov homology and its sister theories, including the homological thinness for alternating links, and questioning if the application of tautological functors for the Khovanov bracket yielded theories that provided additional data on a link. The first conjecture was answered by Eun Soo Lee in her paper [Lee05] and [Lee02], which proved the thinness of alternating knots via the construction of an additional differential on the Khovanov bracket. Jacob Rasmussen [Ras10] furthered this work with by looking at the spectral sequence that arrived out of Lee's endomorphism. In his paper, Jacob Rasmussen constructed the s invariant, which he used to give a combinatorial proof of the Milnor Conjecture.

On a different route, in 2003 Peter Ozsváth and Zoltán Szabó published their paper [OS05] on the relationship between Heegard-Floer Homology of branched double-covers of links, and the Khovanov homology a link. Their relationship showed the existence of a spectral sequence from Khovanov homology to the Heegard Floer homology. This work paved way for a series of papers which asked if Khovanov homology detected the unknot. In 2010, Matthew Hedden and Liam Watson [HW10] determined a large class of knots for which the

Date: January 17, 2014.

Khovanov homology could detect the unknot. Recently, in 2011, Kronheimer and Mrowka published a paper proving that Khovanov homology could detect the unknot. They did this by constructing a new spectral sequence which descended to a new Floer homology theory[KM11]

Rasmussen's s invariant was generalized to a whole family of knot invariants called concordance invariants. In 2005, Cirprian Manolescu and Brendan Owens constructed a concordance invariant for Heegard-Floer homology, called the τ invariant[MO05]. Noticing broad qualities that these theories have in common, Rasmussen in [Ras05] characterized a knot homology theory to be a chain complex whose filtered Euler characteristic categorifies some classical polynomial invariant, and whose filtered chain homotopy type is an invariant, and has a spectral sequence which descends to an invariant which depends only on "coarse" data on the link. In the case of Khovanov and Heegard Floer homology, this "coarse" invariant is the s and τ invariant.

In this paper, I wanted to answer a question about a particular homology theory over $\mathbb{F}_2[u]$ modules introduced in [Bar04]. Turner found several relations between this theory (the Bar-Natan complex) and a filtered theory that was over $\mathbb{F}_2$, and analyzed the spectral sequences that arose out of both theories [Tur04]. I also wanted to look at reduced versions of filtered homology theories to see if the reduced homology theory and the full homology theory extended to the same spectral sequence. Schumakovitch showed in [Shu04] that there is a splitting of the $\mathbb{F}_2$ homology theory into two copies of the reduced homology theory. The natural question to ask in relation to the filtered Bar-Natan theory is if this splitting extends to Turner's spectral sequence.

1.2. **Notation.** Let us set up some notation. In this paper, we will be interested in three homology theories rising from the Khovanov Bracket presented in [Bar04]. For notation and grading conventions we will be following that presented in [Tur06]. In order to describe our homology theories, we will be giving the TQFT associated to them as described in [Bar04].

The (unfiltered) $\mathbb{F}_2$ homology theory arises from the TQFT associated to the Frobenius algebra $V = \mathbb{F}_2[x]/x^2$. The Frobenius algebra over V is equipped with the following multiplication structure

$$m'(1 \otimes 1) = 1 \quad m(1 \otimes x) = m'(x \otimes 1) = x \quad m'(x \otimes x) = 0$$

and an associated comultiplication structure

$$\Delta'(1) = 1 \otimes x + x \otimes 1 \quad \Delta'(x) = x \otimes x$$

By setting the quantum dimension of 1 to q^1, and the quantum dimension of x to be q^{-1} we have that $\dim_q(V) = q + q^{-1}$. In order to make the maps Δ' and m' quantum graded, we will attach a grading shift to the homology. Let v have homological index i and suppose the span of v has q-homogeneous dimension. Then we define the quantum grading of v to be

$$\dim_q(\mathrm{span}(v)) + i$$

Let $C^{\bullet,\bullet}_{\mathbb{F}_2}(D)$ be the $\mathbb{F}_2$ complex associated to a diagram D of a link L. The Khovanov complex is a bigraded theory, with a homological grading arising from the Khovanov bracket, as well as the quantum grading defined above. While $C^{\bullet,\bullet}_{\mathbb{F}_2}$ is technically a cohomology theory, we will call it a homology theory to be consistent with existing literature. As notation, we will always index the homological grading with i, and index the quantum grading with j.

$C^{\bullet,\bullet}_{\mathbb{F}_2}(D)$ is an invariant of the link up to homotopy. In particular, the homology of $C^{\bullet,\bullet}_{\mathbb{F}_2}(D)$ is an invariant of the knot. We denote the homology as $KH^{\bullet,\bullet}_{\mathbb{F}_2}$, and the differential on this complex as d'. Bar-Natan gives an excellent and detailed construction of this homology theory in [Bar02], working with coefficients in $\mathbb{Z}$ instead of $\mathbb{F}_2$.

We construct the $\mathbb{F}_2$ filtered homology theory, which was initially given in Turner's paper. As a $\mathbb{F}_2$ module, $C^{\bullet}_{\mathbb{F}_2}$ is isomorphic to $C^{\bullet,\bullet}_{\mathbb{F}_2}$. However, the TQFT assigns a different differential to this complex, which is defined with the following multiplication and comultiplication maps.

$$\bar{m}(1 \otimes 1) = 1 \quad m(1 \otimes x) = m(x \otimes 1) = x \quad m(x \otimes x) = x$$

$$\Delta(1) = 1 \otimes x + x \otimes 1 + 1 \otimes 1 \quad \Delta(x) = x \otimes x$$

This differential is no longer quantum graded, but it is at least quantum filtered. We denote the filtered theory as $\bar{C}^{\bullet}_{\mathbb{F}_2}$, with homology $\bar{B}N^{\bullet}_{\mathbb{F}_2}$ and differential $\bar{d}$.

Turner's paper looks closely at the $\mathbb{F}_2[u]$ homology theory of a link. While we do not use a TQFT to construct this homology theory, the approach is similar. To each circle we associate the $\mathbb{F}_2$ module $W = \mathbb{F}_2[u][x]/x^2$. Again, the quantum grading of 1 is 1 and the quantum grading of x is -1. The twist on the previous theory is that the variable u is given a quantum grading of -2. This ensures that the following multiplication and comultiplication maps are quantum graded:

$$m(1\otimes 1)=1 \quad m(1\otimes x)=m(x\otimes 1)=x \quad m(x\otimes x)=ux$$
$$\Delta(1)=1\otimes x+x\otimes 1+u1\otimes 1 \quad \Delta(x)=x\otimes x$$

The complex for this homology is again bi-graded, and will be denoted $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}(D)$, and the homology of this complex will be written as $BN^{\bullet,\bullet}(L)$ - the Bar-Natan Theory of a link. While $KH^{\bullet,\bullet}_{\mathbb{F}_2}(L)$ is a 2 indexed object (having homological grading i and quantum grading j), elements in Bar-Natan have 3 gradings: each element has an additional u degree. We will always denote the u degree with the variable k.
As a final piece of notation, we will frequently write $KH^{\bullet,\bullet}_{\mathbb{F}_2}$ for $KH^{\bullet,\bullet}_{\mathbb{F}_2}(L)$ or $KH^{\bullet,\bullet}_{\mathbb{F}_2}(D)$.

1.3. **Known Results.** Without going into detail, here are some known results about $C^{\bullet,\bullet}_{\mathbb{F}_2}$, $\bar{C}^{\bullet}_{\mathbb{F}_2}$, and $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$.

Theorem 1.1 (Turner's Spectral Sequence [Tur04])**.** *There is a spectral sequence from $KH^{\bullet,\bullet}_{\mathbb{F}_2}$ to $\bar{B}N^{\bullet}_{\mathbb{F}_2}$, and the pages of this spectral sequence are link invariants.*

This means that the filtered chain homotopy type of $\bar{C}^{\bullet}_{\mathbb{F}_2}$ is a link invariant.

Theorem 1.2 (Structure of $\bar{B}N^{\bullet}_{\mathbb{F}_2}$ [Tur04])**.** *The dimension of $\bar{K}h^{\bullet}_{F}$ is 2^l, where l is the number of link components.*

These two theorems are a pointer to $\mathbb{F}_2$ being a homology theory in the definition of Rasmussen [Ras05].

Theorem 1.3 (Isomorphism of high u degree homology [Tur04])**.** *For sufficiently large j, there is an isomorphism of chain complexes between $BN^{\bullet,j}$ and $\bar{C}^{\bullet}_{\mathbb{F}_2}$.*

This theorem shows that there is a strong relation between the filtered Bar-Natan theory, and motivates the result of this paper.

Theorem 1.4 (Splitting of $\mathbb{F}_2$ homology into reduced components [Shu04])**.** *$Kh_{\mathbb{F}_2}$ splits into two copies of $\tilde{K}h_{\mathbb{F}_2}$, the reduced Khovanov homology. In particular, there is a exact differential on $Kh_{\mathbb{F}_2}$ of homological degree 0 and u degree 2.*

Schumakovitches result shows that the $\mathbb{F}_2$ reduced theory is equivalent to the $\mathbb{F}_2$ full theory. Turner shows that there exists a similar spectral sequence to that of Theorem 1.1 on the reduced theory. It is natural to ask if this spectral sequence splits in a fashion similar to the unfiltered theory.

1.4. **Summary of Results.** Finally, a summary of the results presented in this paper: We start by giving a presentation for the $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$ that motivates our indexing notation.

Lemma 2.1. There is a bicomplex whose total homology is $BN^{\bullet,\bullet}$

This structure will give us insight into the u torsion of our homology theory

Theorem 3. Let $v\in BN^{\bullet,\bullet}$. Suppose that Turner's spectral sequence converges after the m page. If $u^{m-1}v\neq 0$ then $u^n v\neq 0$ for all $n\geq m-1$

Corollary 3.2. Suppose that the spectral sequence for computing Bar-Natan theory converges after m pages. Elements with u degree greater than m are wholly determined by $BN^{\bullet,\bullet}_{\infty}$

We attempt to extend Schumakovitch's result [Shu04] on the reduced homology theory over $\mathbb{F}_2$ to the reduced homology theory over $\mathbb{F}_2[u]$.

Theorem 4.6. There is a differential s^* on $BN^{\bullet,\bullet}$ that shifts quantum degree by 2 and is exact on homology.

Finally, we give an explicit equivalence between the $\mathbb{F}_2[u]$ homology and the pairings given by the filtered theory.

Theorem 6.4. The number of elements in $BN^{\bullet,\bullet}$ that only have representatives that are filtered in u degree is no more than 2^l.

Theorem 6.6. There is an equivalence of data between the pairing data given by Turner's spectral sequence and the u torsion in $BN^{\bullet,\bullet}$.

What we show is an equivalence of categories: the homotopy category of chain complexes over $\mathbb{F}_2[u]$, and the filtered homotopy category over $\mathbb{F}_2$.

2. The Bar Natan Bicomplex

While $\mathbb{F}_2[u]$ theory has the advantage of being a graded theory, the u grading is artificial in nature: in fact, the most natural way to compute $\mathbb{F}_2[u]$ homology is to use a spectral sequence filtered in the u degree. In this section, we give an alternative definition for the $\mathbb{F}_2[u]$ homology.
As $\mathbb{F}_2[u]$ theory is graded, it is natural to split it across its quantum grading. We can write

$$C^{\bullet,\bullet}_{\mathbb{F}_2[u]} = \bigoplus_i C^{\bullet,j}_{\mathbb{F}_2[u]}$$

This simplification allows us to compute each subcomplex $C^{\bullet,j}_{\mathbb{F}_2[u]}$ separately. However, these subcomplexes can be difficult to compute themselves, as the maps they contain are only u filtered, as oppose to being u graded. One solution to computing the homology of $C^{\bullet,j}_{\mathbb{F}_2[u]}$ is to use a u filtered spectral sequence (as given by Turner [Tur04]). We propose a different way to view that $C^{\bullet,j}_{\mathbb{F}_2[u]}$, which showcases the fact that the differential only increases the filtration by at most 1.

Definition 2.1. Let $BC^{k,l}_j[L]$ be the $\mathbb{F}_2$ module generated by elements of $C^{\bullet,j}_{\mathbb{F}_2[u]}$ with homogeneous u grading l and homological index $k+l$. Equip $BC^{\bullet,\bullet}_j[L]$ with a bicomplex structure with these two differentials:

- A differential with (k,l) degree $(0,1)$ corresponding to the differential d' from unfiltered $\mathbb{F}_2$ theory.
- A differential Φ with (k,l) degree $(1,0)$ corresponding to new multiplication and comultiplication maps:

$$m_\Phi(1\otimes 1) = 0 \quad m_\Phi(1\otimes x) = m_\Phi(x\otimes 1) = 0 \quad m_\Phi(x\otimes x) = ux$$
$$\Delta_\Phi(1) = u1\otimes 1 \quad \Delta_\Phi(x) = 0$$

We call this $\mathbb{F}_2$ bicomplex the **Bar-Natan Bicomplex** of a link L.

Keeping track of all of the indices is a pain, so here is a quick reference:

$$BC^{u\text{-grading, (homological grading -}u\text{ grading)}}_{\text{Quantum Grading}}$$

The commutativity of d' and ϕ is clear from checking that $d'\phi - \phi d' = 0$.

Lemma 2.1. *The total homology of the $BC^{k,l}_j$ is $BN^{\bullet,j}$*

Proof. The total homology of a bicomplex is given by objects $Tot(BC^{k,l}_j) = \bigoplus_{k+l=i} E^{k+,l}_{j,0}$ and is endowed with the differential $d_{tot} = d'+\phi$. From our original construction of the spectral sequence, $\bigoplus_{k+l=i} E^{k,l}_{j,0} = C^{i,j}$ and the differential $d = d'+\phi = d_{tot}$. □

With this language, instead of treating the complex as having a filtration, we can think of the u as an additional index. There is also a nice way to visualize this theory. One can think of this theory as a stack of "sheets", with each sheet corresponding to one quantum grading 1. Note that this theory only has support for positive l, and as $C_{\mathbb{F}_2[u]}$ only has support of every other quantum grading, $BC^{k,l}_j$ only exists for every other j
With this notation we see that there are a number of different ways to compute the homology of the this theory using spectral sequences. While we take homology with respect to the homological index first, or with respect to the u index first, taking homology with respect to the homological index is more useful:

Lemma 2.2 (Turner's Spectral Sequence [Tur04]). *Let $E_0, E_1, E_2\dots$ be a spectral sequence that computes the total homology of $C^{\bullet,j}_{\mathbb{F}_2[u]}$ by first taking a differential of k degree 1. Then $E_1, E_2, \dots$ are link invariants.*

Proof. As taking a differential first in the k direction corresponds to taking the differential d', we see that the E_1 page is (as a module) the sum of many copies of $KH^{\bullet,\bullet}$. It follows that the higher pages are link invariants. □

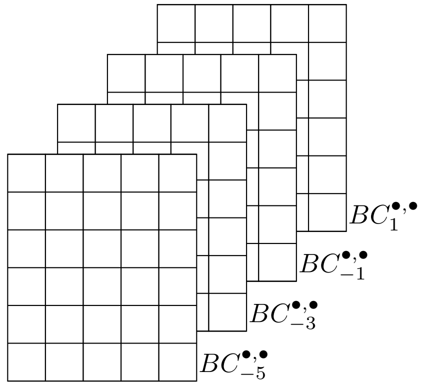

FIGURE 1. A stack of $BC_j^{\bullet,\bullet}$ makes up $C_{\mathbb{F}_2[u]}^{\bullet,\bullet}$

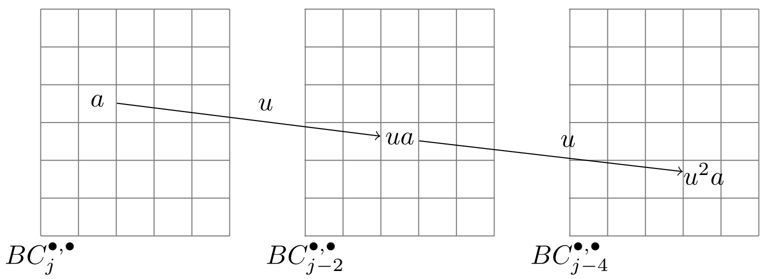

FIGURE 2. Multiplication by u map between bicomplexes

We may have well taken the spectral sequence along the other index. This spectral sequence would be k filtered instead, from computing the l-indexed homology first. While the pages of this spectral sequence are not invariants of the link, one can use both spectral sequence as additional information to aid the computation of the total homology.

2.1. **Multiplication by u.** The action of u on the complex $C_{\mathbb{F}_2[u]}^{\bullet,\bullet}$ is a chain endomorphism. It therefore induces a map u^* on the homology $BN^{\bullet,\bullet}$. As multiplication by u is shifts quantum grading by -2, we can think of this as a map

$$u : BC_j^{\bullet,\bullet} \to BC_{j-2}^{\bullet,\bullet}$$

which shifts entries down along the diagonal. This is perhaps best seen in Figure 2. The Bicomplex structure of Bar-Natan theory gives us some insight into how u multiplication should work. We first notice that $BC_{j-2}^{\bullet,\bullet}$ is a brutal truncation of $BC_j^{\bullet,\bullet}$ in the l (u-indexed) direction. This tells us indication that the total homology of $BC_{j-2}^{\bullet,\bullet}$ and $BC_j^{\bullet,\bullet}$ only differ by a small amount, and that difference should encode data on how we cut our complex apart. In fact, for large enough j, we expect the homology to be an isomorphism.

Lemma 2.3 (Stable Bar-Natan Theory [Tur04]). *There exists j' such that for all $j'' > j'$, we have that $BC_{j'}^{\bullet,\bullet} \cong BC_{j''}^{\bullet,\bullet}$*

Proof. The finiteness of support of the Khovanov complex implies that for sufficiently large j, the brutal truncation taking $BC_{j''}^{\bullet,\bullet} \to BC_{j'}^{\bullet,\bullet}$ is an isomorphism. $\square$

Corollary 2.4 (Turner [Tur04]). *There exists i' such that for all $j'' > j'$, we have that $BN^{\bullet,j'} \cong BC^{\bullet,j''}$*

We abuse notation and call this particular bicomplex $BC_\infty^{\bullet,\bullet}$ with homology $BN^{\bullet,\infty}$. When we write $BC_\infty^{\bullet,\bullet}$ we mean the smallest i such that $BC_i^{\bullet,\bullet}$ is the full complex. What this isomorphism (at large u degree) of multiplication by u shows is the obstruction for multiplication by u to being a isomorphism of complexes is the truncation of the bicomplex along the u index.

Theorem 2.5. *Let $[v] \in BN^{\bullet,\bullet}$. Suppose that Turner's spectral sequence converges after the m page. If $u^{m-1}[v] \neq [0]$ then $u^n[v] \neq [0]$ for all $n \geq m-1$*

Proof. Suppose that $[v] \in BN^{\bullet,\bullet}$ has non trivial image under multiplication by u^{m-1}. Then we have that $u^{m-1}v \in \ker d$, but $u^{m-1}v \notin \operatorname{Im} d$. We now want to show that $u^m v$ survives in the homology.
As $d(u^m v) = ud(u^{m-1}v) = u0 = 0$, we have that $u^m v$ is in the kernel of d. Now it remains to show that $u^m v$ is not in the image of d. Suppose for contradiction that there exists w such that $d(w) = u^m v$. We have that the minimal u degree of w must be 0, because there is no w' such that $d(uw) = u^m v$. Therefore, there must be an induced differential between 0 degree u-homogeneous part of w and $u^m v$. Of course, this induced differential must be of u grading m. But there are no induced differentials of u filtration m.
Repetition of this argument shows that $u^n v$ is in the homology for all $n > m$ $\square$

Recall, we call a link $\mathbb{F}_2$ thin if $KH_{\mathbb{F}_2}$ is supported on two diagonals.

Corollary 2.6. *Suppose that a link is $\mathbb{F}_2$ thin. Then every element in $BN^{\bullet,\bullet}$ is either u torsion free or has u torsion 1.*

Proof. This follows from Turner's result [Tur04] that $\mathbb{F}_2$ thin knots converge on the E_2 page, and Theorem 3 $\square$

These theorems tell us pictorially how our homology should look.

Lemma 2.7. *Suppose that $u[v] \neq [0]$ in $BN^{\bullet,\bullet}$. Then $[v] \neq [0]$.*

Proof. This proof simply follows from the fact that multiplication by u is like moving the brutal truncation of $BC_j^{\bullet,\bullet}$ Since $d(uv) = ud(v) = 0$, it is easy to see that $v \in \ker d$. It remains to show that it is not in the image of d.
Suppose that there did exist w such that $d(w) = v$. Then $d(uw) = d(uv)$, which would imply that $[uw] = 0$, contradicting the nontriviality of $[uv]$. $\square$

$BN^{\bullet,\bullet}$ theory can be decomposed into a **band of u-torsion homology** and a **tower of u-stable homology**. Lemma 2.7 shows that there is a set elements $\{v_a\}$ (not necessarily u homogeneous) which have the following properties

- They have a term of u degree 0
- The classes $[v_a], [uv_a], [u^2v_a], \dots$ generate $BN^{\bullet,\bullet}$

Later we will strengthen this result for the band of the homology.

3. Computing $BC_\infty^{\bullet,\bullet}$

To get some additional structure on Bar-Natan theory, we look at the structure for high quantum degree. A lot of the work in this section follows that of Turner, who has determined the dimension of high quantum degree Bar Natan theory by finding an isomorphism to the filtered theory. Here, we compute the dimension. For this section, we return to thinking of $BN^{\bullet,\bullet}$ as a module theory, rather than a bicomplex theory.

We pick a new basis for Bar Natan complex. While the usual basis of

$$1, x, u1, ux, u^2 1, u^2 x \dots$$

for V is good in that the elements are homogeneous in u degree, we want a basis which plays nice with the differential of complex. Turner [Tur04] uses the basis $1+x, x$ for computing the filtered Bar-Natan theory: this basis has the pleasant property of diagonalizing the differential on $\overline{C}_{\mathbb{F}_2}^{\bullet,\bullet}$, but it has the unpleasant property of be nonhomogenous in quantum degree. Fortunately, in the $\mathbb{F}_2[u]$ theory we can find a basis which almost diagonalizes the differential, and is homogeneous in quantum grading.
Consider the elements $a = u1 + x$, $b = x$.

$$1, a, b, ua, ub, u^2a, u^2b \dots$$

form a basis for $\mathbb{F}_2[u]\{1, x\}$ as before. This basis has two important properties. First off, it diagonalizes the differential wherever the u grading is greater than 1. Secondly, the basis is homogeneous in quantum degree. We observe that multiplication and comultiplication now have the following form:

-1					
-3					
-5	1	1			
-7		2	2		
-9			1	1	
-11					

-1					
-3					
-5	1				
-7		1	1		
-9				1	
-11					

-1					
-3					
-5	1				
-7					
-9				1	
-11					

-1					
-3					
-5	1				
-7					
-9					
-11					

FIGURE 3. A possible spectral sequence

$$m(a\otimes a)=ua \quad m(a\otimes b)=0 \quad m(b\otimes b)=ub$$
$$\Delta(a)=ua\otimes a \quad \Delta(b)=b\otimes b$$

Note that in all cases but co-multiplication of b, the differential raises the u degree of the basis element by one. This is an easy way to see that the differential on the Bar-Natan complex splits as $d=d'+\phi$, where the u degree of d' is 0, and the u degree of ϕ is 1.

From here we use a technique of Lee to show that the generators for homology must be fairly simple. We construct an inner product on the basis a,b for the homology, by letting a,b be an orthonormal basis. We then define a dual differential on the complex, d^* which is given by multiplication and comultiplication maps

$$\Delta^*(a\otimes a)=ua$$
$$\Delta^*(b\otimes b)=b$$
$$\Delta^*(a\otimes b)=0$$
$$m^*(a)=ua\otimes a$$
$$m^*(b)=b\otimes b$$

A result of a complex equipped with a differential that is diagonal with respect to the inner product is that the homology at a point is given by the intersection of kernels and dual kernels.

Theorem 3.1 (Turner [Tur04]). *The dimension of $BN^{\bullet,\infty}_{\mathbb{F}_2[u]}$ as a $\mathbb{F}_2$ module is 2^l, where l denotes the number of link components.*

Proof. Once we look at a sufficiently high quantum degree, the basis for the chain complex $C^{\bullet,\infty}_{\mathbb{F}_2[u]}$ is given completely by terms of the form $u^n a, u^n b$, a basis that diagonalizes the differential. We should expect the homology to be relatively simple for large u degree. We follow Lee's proof [Lee02] to show that the homology is of degree 2. In fact, we can explicitly provide generators for the homology. We take a smoothing and label the circles in the smoothings a or b. This gives us an element in the chain complex. If every pair of circles that share crossing have different labels, then the corresponding element in the chain complex lies in the homology, as the application of the multiplication or comultiplication map will always take a pair of different labellings to 0. Lee shows that the only smoothings that have this property are those corresponding to a resolution of the knot in an orientation preserving fashion. For a link, the number of such resolutions is 2^l, where l is the number of link components. From here, Lee exhibits two steps to compute the homology. First, we show that there are labeled states that have the property above. Then we want to show that these states are sufficient for generating the whole homology. □

When the theory is plotted on a grid with homological index on one axis, and the quantum grading on the vertical axis, the places where these 2^l generators take support make a tower of homology. We will refer to it as such from here on out.

The structure of $BN^{\bullet,\infty}_{\mathbb{F}_2[u]}$ tells us that the differential d is almost exact for large u degree. This shows that the spectral sequence computing $BN^{\bullet,\infty}_{\mathbb{F}_2[u]}$ induces a pairing on the elements on $BC^{\bullet,\bullet}_{\infty}$ except on 2^l elements. Two elements are paired if in the spectral sequence they are connected by an induced differential.

Corollary 3.2. *Suppose that the spectral sequence for computing Bar-Natan theory converges after m pages. Elements with u degree greater than m are wholly determined by $BN^{\bullet,\infty}$*

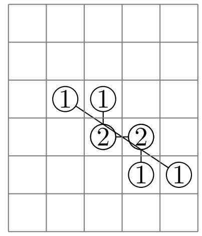

FIGURE 4. Pairing elements in the spectral sequence

Proof. We have by Theorem we have that if $[v]$ has support in homology, then $[u^m v]$ has support in homology for all m greater than n. □

4. Reduced Bar-Natan Theory

In this section, we aim to extend Schumakovitch's result on the reduced $\mathbb{F}_2$ homology to the $\mathbb{F}_2[u]$ homology.

Theorem 4.1 (Schumakovitch [Shu04])**.** *The $\mathbb{F}_2$ homology theory splits as two copies of the reduced theory*

Recall that the reduced $\mathbb{F}_2[u]$ theory is given by the subcomplex where a specially noted circle in the state diagram is always given a x marking [Kho02]. For simplicity, when we will place a dot over the entry that is marked. We will denote this subcomplex and theory $\widetilde{C^{\bullet,\bullet}_{\mathbb{F}_2[u]}}$ and $\widetilde{BN^{\bullet,\bullet}}$ respectively.

Theorem 4.2. *Suppose that the spectral sequence collapses on the E_2 page. Then the u torsion section of the $\mathbb{F}_2[u]$ homology splits into two copies of the reduced theory, $\widetilde{BN^{\bullet,\bullet}}$*

We follow the footsteps of Schumakovitch's proof. We first define a chain map

$$s : C^{\bullet,\bullet}_{\mathbb{F}_2[u]} \to C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$$

which decreases quantum grading by two. Let $v \in C^{\bullet,\bullet}$ be a state. Then let V_k be the set of all states where x has been replaced by 1 in v a total of k times. Then define

$$s(v) = \sum_{k=1}^{\infty}(u^{k-1} \sum_{w \in V_k} w)$$

For example,

$$\begin{aligned} s(1 \otimes x \otimes x \otimes x) =& 1 \otimes 1 \otimes x \otimes x + 1 \otimes x \otimes 1 \otimes x + 1 \otimes x \otimes x \otimes 1 \\ &+ u(1 \otimes 1 \otimes 1 \otimes x + 1 \otimes 1 \otimes x \otimes 1 + 1 \otimes x \otimes 1 \otimes 1) \\ &+ u^2(1 \otimes 1 \otimes 1 \otimes 1) \end{aligned}$$

We can alternatively define s recursivly on v. We first break into subcases on the length of v, then on the leading letter.

(1) Suppose that v has length greater than 1. Then we write $v = a \otimes w$, where $a = 1$ or $a = x$.
   (a) If $a = 1$, then $s(v) = 1 \otimes s(w)$

   (b) If $a = x$, then $s(v) = u1 \otimes s(w) + 1 \otimes w + x \otimes s(w)$
(2) Suppose that v has length one. Then we have that $s(1) = 0$, and $s(x) = 1$

Lemma 4.3. *The map s commutes with the differential on $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$*

Proof. There is just some bookkeeping to do here. We will show it for the most complicated case. Let m be the multiplication map which acts on the first two entries of a state $v = x \otimes x \otimes w$, where w is the remainder of the state v.

$$\begin{aligned} s(m(x \otimes x \otimes w)) =& s(ux \otimes w) \\ =& u(1 \otimes w + x \otimes s(w) + u * 1 \otimes s(w)) \end{aligned}$$

$$\begin{aligned} m(s(x \otimes x \otimes w)) =& m(u^2 1 \otimes 1 \otimes s(w)) \\ &+ m(u(1 \otimes x \otimes s(w) + x \otimes 1 \otimes s(w) + 1 \otimes 1 \otimes w)) \\ &+ m(x \otimes x \otimes s(w) + 1 \otimes x \otimes w + x \otimes 1 \otimes w)) \\ =& u(1 \otimes w + x \otimes s(w) + u * 1 \otimes s(w)) \end{aligned}$$

□

There is an additional map $k : C^{\bullet,\bullet}_{\mathbb{F}_2[u]} \to C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$ which increases the u grading by 2, and commutes with the differential. The map k takes

$$\dot{1} \mapsto \dot{x}$$

and

$$\dot{x} \mapsto \dot{u}x$$

Lemma 4.4. *k commutes with the differential on $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$*

Proof. Again, we show the most difficult case

$$\begin{aligned} k(\Delta(\dot{1} \otimes w)) =& k(\dot{1} \otimes x \otimes w + \dot{x} \otimes 1 \otimes w + u \cdot \dot{1} \otimes 1) \\ =& \dot{x} \otimes x \otimes w + u \cdot \dot{x} \otimes 1 \otimes w + u \cdot \dot{x} \otimes 1 \otimes w \\ =& \Delta(k(\dot{1} \otimes w)) \end{aligned}$$

□

As both s and k commute with the differential, they descend to maps on the homology, $s^* : BN^{\bullet,\bullet} \to BN^{\bullet,\bullet}$ and $k^* : BN^{\bullet,\bullet} \to BN^{\bullet,\bullet}$.

Lemma 4.5. *s is acyclic on $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$*

Proof. There is a combinatorial proof of this fact, which is not revealing of the structure of homology. We give that proof now, and save the topological proof until the section on tautological functors. □

Proof. We want to show that $s(s(v)) = 0$ where v is some state generating $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$. Let m be the number of x's that appear in v. Then we have

$$\begin{aligned}
s(s(v)) &= s(\sum_{k=1}^{\infty}(u^{k-1}\sum_{w\in V_k} w)) \\
&= \sum_{k=1}^{m}(u^{k-1}\sum_{w\in V_k} s(w)) \\
&= \sum_{k=1}^{m}(u^{k-1}\sum_{w_\alpha\in V_k}(\sum_{j=1}^{m-k}(u^{j-1}\sum_{y\in W_j^\alpha} y)) \\
&= \sum_{l=2}^{m}\sum_{k+j=l} u^l \sum_{w_\alpha\in V_k}\sum_{y\in W_j^\alpha} y \\
&= \sum_{l=2}^{m} u^{l-1}\left(\sum_{\substack{k+j=l\\k<j}}\left(\sum_{w_\alpha\in V_k}\sum_{y\in W_j^\alpha} y\right) + \sum_{\substack{k+j=l\\k>j}}\left(\sum_{w_\alpha\in V_k}\sum_{y\in W_j^\alpha} y\right) + \sum_{\substack{k+j=l\\k=j}}\left(\sum_{w_\alpha\in V_k}\sum_{y\in W_j^\alpha} y\right)\right) \\
&= \sum_{l=2}^{m} u^{l-1}\left(\sum_{\substack{k+j=l\\k<j}}\left(\sum_{w_\alpha\in V_k}\sum_{y\in W_j^\alpha} y\right) + \sum_{\substack{k+j=l\\k<j}}\left(\sum_{w_\alpha\in V_j}\sum_{y\in W_k^\alpha} y\right) + \sum_{\substack{k+j=l\\k=j}}\left(\sum_{w_\alpha\in V_k}\sum_{y\in W_j^\alpha} y\right)\right) \\
&= \sum_{l=2}^{m} u^{l-1}\left(0 + \sum_{\substack{k+j=l\\k=j}}\left(\sum_{w_\alpha\in V_k}\sum_{y\in W_k^\alpha} y\right)\right)
\end{aligned}$$

These middle terms also match up, as there is a pairing between W_k^α and $W_k^{\bar\alpha}$, where $w_{\bar\alpha}$ has 1 where w_α has a x marked. This shows that s is an exact function. □

Theorem 4.6. *The function s^* is acyclic on $BN^{\bullet,\bullet}$*

Proof. Mirroring the work in Schumakovitche's paper, we start by showing that $sk + ks = 1 + us$. Let $v = \dot{a}\otimes w$ be a state in $C^{\bullet,\bullet}_{\mathbb{F}_2[u]}$, where the dot marks the specially marked cycle. We break into 2 cases.

(1) $a = 1$. Then

$$sk(v) = s(\dot{x}\otimes w) = u\dot{1}\otimes s(w) + \dot{x}\otimes s(w) + \dot{1}\otimes w$$

and

$$ks(\dot{1}\otimes w) = k(\dot{1}\otimes s(w)) = \dot{x}\otimes s(w)$$

. Therefore,

$$\begin{aligned}
(sk+ks)(v) &= 1\otimes w + u1\otimes s(w) \\
&= (1+us)(v)
\end{aligned}$$

(2) $a = x$. The

$$sk(v) = s(u\dot{x}\otimes w) = u^2\dot{1}\otimes s(w) + u\dot{x}\otimes s(w) + u\dot{1}\otimes w$$

and

$$\begin{aligned}
ks(x\otimes w) &= k(x\otimes s(w) + u1\otimes s(w) + s(w) + 1\otimes w) \\
&= u\dot{x}\otimes s(w) + u\dot{x}\otimes s(w) + x\otimes w
\end{aligned}$$

Bringing

$$\begin{aligned}
(sk+ks)(v) &= u^2\dot{1}\otimes s(w) + u\dot{x}\otimes s(w) + u\dot{1}\otimes w + x\otimes w \\
&= (1+us)(v)
\end{aligned}$$

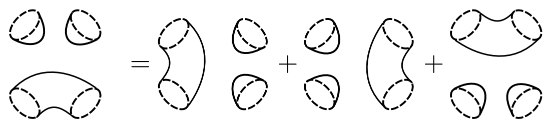

Figure 5. The $4-tu$ relationship over $\mathbb{F}_2$

Now to show that s^* is acyclic on $BN^{\bullet,\bullet}$. As $s^2=0$, we need only show that the kernel of s is contained in the image of s to have exactness. Suppose $s^*([v])=0$. Then we have that $s^*k^*(v)=(1+u^*s^*+k^*s^*)([v])=[v]$. Therefore, s is exact. □

Corollary 4.7. *Suppose that $BN^{\bullet,\bullet}$ converges on the second page of Turner's spectral sequence. Then the map k^* is acyclic on the torsion band of homology.*

Proof. There is some difficulty here as $k^2\neq 0$. However, $(k^*)^2=u^*k^*$ and if $BN^{\bullet,\bullet}$ converges on the second page, u^* is trivial on homology. Therefore, $(k^*)^2=0$.
We need now only show that if $k^*([v])=[0]$, then there exists $[w]$ such that $k([w])=[v]$. By corollary 2.6 multiplication by u is trivial on the band of torsion in $\mathbb{F}_2$ thin links, so we have that $k^*s^*([v])=(1+u^*s^*+k^*s^*)=[v]$ □

This corollary brings up an interesting point. In regular $\mathbb{F}_2$ homology, we had that the both k and s were exact maps. Now, the exactness of k depends on the u torsion of the complex. Without the exactness of k, analyzing the reduced Bar-Natan theory is difficult. We therefore take a look at the image of s instead, which forms a subcomplex of $BN^{\bullet,\bullet}$. We will call this object the **top reduced Bar-Natan theory**, written as $\widetilde{C^{\bullet,\bullet}_{\mathbb{F}_2[u]}}^{T}$.

5. Cobordism representation of Schumakovitch's Map

Can we construct the s and t maps given by Schumakovitch for arbitrary homology theories? We want to find under what conditions we can have an exact differential between the full homology theory and the reduced homology theory. To do this we need to look at Bar Natan's original description of the $\mathbb{F}_2[u]$ theory. In his paper on Cobordisms [Bar04], Bar-Natan constructed the $\mathbb{F}_2[u]$ theory as a purely cobordism-based theory. Recall, the category $\mathcal{C}ob^3{}_l$ is the category of $1+1$ cobordisms, with the additional relationships of S, T, and $4-tu$. From here, Bar-Natan considers a different category, $\mathcal{BN}$. We characterize this category by giving it's set of objects and morphisms.

- An object in $\mathcal{BN}$ is the set of morphisms that start at a single circle and end at some set of circles, $L_\alpha\in Ob(\mathcal{C}ob^3{}_l)$. The objects are written as $Hom_{\mathcal{C}ob^3{}_l}(\circ,L_\alpha)$. Remember we take the cobordisms and have modded out by the relationships $l=\{S,T,4-tu\}$. We further tensor by $\mathbb{F}_2$, thereby removing all 2 torsion from out theory. The $4-tu$ relationship, along with the lack of 2 torsions, means that there is a canonical representation for a cobordism under the relationship l. The canonical form of a cobordism is where every connected component save one is a disk. The last component is a punctured n-torus, where one puncture corresponds to the source of the cobordism. We can now name every cobordism. We write a 1 for every boundary the belongs to a disk, and an x for every boundary that belongs to the special component. Finally, we write u^n to denote the number of "donut holes" in the special component. In such as fashion, we can derive a "name" for every cobordism. See the above figure for an example.

- For morphisms, $\mathcal{BN}$ uses the morphisms from $\mathcal{C}ob^3{}_L$, where the action between $Hom_{\mathcal{C}ob^3}(\circ,L_\alpha)$ and $Hom_{\mathcal{C}ob^3}(\circ,L_\beta)$ is given by the pullback.

This theory gives the same homology as the $\mathbb{F}_2[u]$ theory. Let $\mathcal{F}$ be the functor that takes an object $Hom_{\mathcal{C}ob^3{}_l}(\circ,L_\alpha)$ and in this theory, we can provide a natural rendition of Schumakovitch's s map.

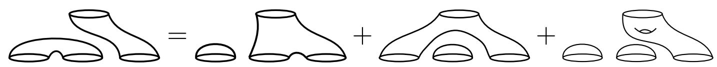

FIGURE 6. Reduction of a cobordism to the standard form

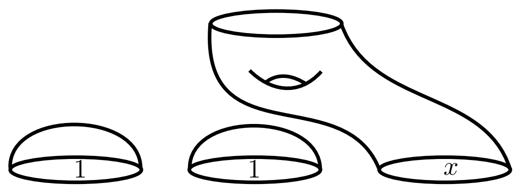

FIGURE 7. The cobordism representing $u1 \otimes 1 \otimes x$

Theorem 5.1. *Consider the function s' taking a cobordism $C : \circ \to L_\alpha$ (in the canonical form) to the new cobordism where we have inserted a cut close to the domain of the cobordism. This map is the same as Schumakovitch's differential s.*

Proof. We show that this function follows the recursive definition for Schumakovitch's map. We first choose a representative for the cobordism that is in the canonical form. We now analyze some subcases:

- The special component has only one boundary component (and therefore has labeling $1 \otimes 1 \otimes \ldots \otimes 1$.) In this case, cutting C anywhere near the domain creates (up to the relationships in l) the cobordism $C \cup S^2$, the two sphere. However, the two sphere is 0 under the relationship S, so $s'(C) = 0$
- The special component has at least two boundary components. Then the cutting s' creates a cobordism with special component with boundary the domain, unchanged punctured spheres, and lower component with is a punctured sphere whose boundary completely lies in the codomain of the cobordism.

In the first case, we have a cobordism that agrees with Schumakovitch's map. It therefore suffices to check the second case. Use the $4 - tu$ relationship on a handle in the lower component, and the special component. This sum of cobordisms that results corresponds to the same sum of states given by Schumakovitch's map. See Figure 8. □

The cobordism description of the s map makes it quite clear why s is a map of the chain complexes: commutativity with the differential is obvious in the cobordism description, as the neck cutting relationship occurs far away from where morphisms are being composed.

Corollary 5.2. *s is a differential on the $\mathbb{F}_2[u]$ complex*

Proof. In the cobordism theory, s^2 is applying two cuts near the special top component. However, after the application of a single cut, the top component is a disk. The second cut "lops" of a sphere from this specially marked component. According to the relationship S, this corresponds to multiplication by 0. Therefore, $s^2 = 0$ □

There is a similar description of Khovanov's k map using cobordisms. Consider the map k' that connects a marked boundary of the cobordism to the special component with a handle. It is easy to verify that this map is the same as the map on $\mathbb{F}_2[u]$ modules, k.
It should be noted that if we take the relationship that $T = 0$ in the cobordism theory, we get the original $\mathbb{F}_2$ homology. In this context, the cutting map given by s' and the handle-adding map given by k' are exactly the same as the ones used by Schumakovitch to prove the exact sequence on $\mathbb{F}_2$ homology. Exactness of the map k' follows as the addition of two handles to a surface corresponds to adding a donut hole, and we have that $T = 0$.
Finally, if we take the relationship $T = 1$ in the cobordism theory, we get the filtered $\mathbb{F}_2$ homology, and a new type of Schumakovitch map.

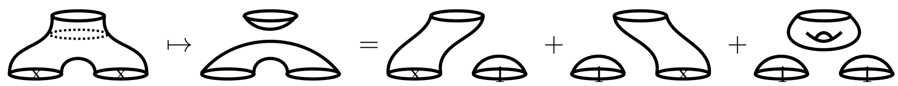

Figure 8. The cutting of a cobordism yields Schumakovitch's map

From here, we see the exact requirements that a homology theory needs in order to have an exact differential between the full and reduced theories, and why the $\mathbb{F}_2[u]$ theory does not posses such a differential. It is interesting to note that the differential s is always exact on homology.

6. Relationships between Reduced, Filtered and $\mathbb{F}_2[u]$ theories

In this section, we give equivelence required to reconstruct the $\mathbb{F}_2$ spectral sequence from just the module theory. To avoid mucking around with indices, we will instead show that the spectral sequence for $BC_\infty^{\bullet,\bullet}$ contains all the information of $BN^{\bullet,\bullet}$, and vice versa. We will then use a map between $BC_\infty^{\bullet,\bullet}$ and $C_{\mathbb{F}_2}^{\bullet,\bullet}$ to determine the information on the filtered theory.

Lemma 6.1 (Turner [Tur04])**.** *There exists a natural projection π from $BC_\infty^{\bullet,\bullet}$ to Turner's filtered theory.*

The u degree will give us information on how to construct the higher differentials. In return, the filtered theory will give us some information on the u non-homogenous generators of the $\mathbb{F}_2[u]$ theory. Recall that Bar-Natan theory, while quantum graded, is only u filtered.

6.1. **Filtered Width.** Suppose that there is an element $\bar{v}$ in the Bar-Natan homology that does not have a representative in $C_{\mathbb{F}_2[u]}^{\bullet,\bullet}$ which is u homogeneous. Then let v be an representative of that class in $C_{\mathbb{F}_2[u]}^{\bullet,\bullet}$. We have that v can be written as a sum of elements with homogeneous u degree because as a module $C_{\mathbb{F}_2[u]}^{\bullet,\bullet}$ splits across u degree. Write $v = \sum_{i_0 \leq i \leq i_f} v_i$, where each v_i is u-homogeneous of degree i. We call $|i_f - i_0|$ the **filtered width** of v and write is $\omega(v)$

Definition 6.1. We call v **primitive** if there does not exist w, t such that $[w] + [t] = [v]$ and $\omega(w), \omega(t) < \omega(v)$.

Lemma 6.2. *If v is primitive, then no $v_j = 0$.*

Proof. Suppose that there was some v_j such that $v_j = 0$. Then we could write $v = \sum_{i_0 \leq a < j} v_a + \sum_{j < b < i_f} v_b$. Let $w = \sum_{i_0 \leq a < j} v_a$, and $t = \sum_{j < b < i_f} v_b$. Then the maximal u degree of w and the minimal u degree of t differ by 2. In particular, we have that the maximal u degree of $d(w)$ and minimal u degree of $d(t)$ differ by 1. So both w and t lie in the kernel of d, and therefore represent an element in the homology. But this contradicts the primitiveness of w. □

Lemma 6.3. *Let v be primitive. Then we have that $\phi(v_j) \neq 0$ for all $j \neq i_0$, and we have that $d'(v_j) \neq 0$ for all $j \neq i_f$.*

Proof. If $d'(v_j) = 0$ for some j, then we have that $\phi(v_{j-1}) = 0$ as $d'(v_i) + \phi(v_{i-1}) = 0$. We now split $v = \sum_{i_0 \leq a < i} v_a + \sum_{j \leq b < i_f} v_b$. Let $w = \sum_{i_0 \leq a < j} v_a$, and $t = \sum_{j \leq b < i_f} v_b$. Then $d(a) = d(b) = 0$. But this again contradicts the primitivity of w. □

Theorem 6.4. *If v is primitive with filtered width greater than 0, then $[uv] \neq [0]$ on homology.*

Proof. We have that $d(uv) = ud(v) = 0$, so it remains to show that uv is not in the image of d. Suppose there is w such that $d(w) = uv$. We write $w = \sum_{k_0 \leq k \leq k}$. Since uv is in the image of k, we have for every v_i, that $uv_i = d'(w_{i+1}) + \phi(w_i)$. Look at v_{i_0}. We know that $\phi(uv_{i_0}) \neq 0$ by the previous lemma. As ϕ is a chain map, this means that there is no w_{i-1} such that $\phi(w_i) = uv_{i_0}$. This means that w is completely supported in u degrees greater or equal to $i_0 + 1$. However, this means that the u degree of every w_k is greater that 1. Let $u^{-1}w$ be the element that maps to w under multiplication under u. Then $d(u^{-1}w) = v$, so $[v] = [0]$. This contradicts the filtered width of v. □

Corollary 6.5. *The dimension of the space spanned by elements of filtered width greater than* 0 *in $BN^{\bullet,j}$ is no more than 2^l, where l is the number of link components.*

Proof. As elements with filtered width greater than 0 are u torsion free, we can see by the large quantum degree theory that the number of filtered elements must be 2^l per quantum degree. □

6.2. **u-torsion and Higher Differentials.** In this section, we show how knowing the u torsion of an element in the homology gives information on higher differentials in the filtered theory of $BN^{\bullet,\infty}$. In fact, we would like to show that the data from these two theories are equivalent.

Theorem 6.6. *The spectral sequence convereging to $BN^{\bullet,\infty}$ and the u torsion of elements in $BN^{\bullet,\bullet}$ determine eachother completely.*

We've seen before that the convergence rate of Turner's spectral sequence provides an upper bound for the u torsion of the band of homology. We now want to show that the u torsion of an element determines which pages it shows up on in Turner's spectral sequence for filtered $\mathbb{F}_2$ theory.
We first show that high u torsion implies the existence of a high u degree differential in the spectral sequence. When we write v has u degree 1, we mean to pick an element v in the band of u torsion.

Lemma 6.7. *Suppose that v is of u degree 0 and $[v]$ has u torsion n in the homology. Then $\pi(v)$ is on the $n+1$ page of Turner's spectral sequence, and is not on the $n+2$ page of Turner's spectral sequence.*

Proof. Let $d(w) = u^n v$, and let $u^n v$ live in $BN^{\bullet,j}$. Then write $w = \sum_{l=0}^{\infty} w_l$, where w_l is u-homogenous of degree l. We have that $w_0 \neq 0$, as if w had no support in u degree 0, the torsion of v would be $n-1$. We claim that $d'(w_k) \neq 0$ for all $k \neq 0$ and $\phi(w_k) \neq 0$ for any $k < n$. Suppose that there existed w_l such that $d'(w_k) = 0$. Then consider the element $\sum_{l \geq k} w_l$. We have that $d'(\sum_{l\geq k} w_l) = u^n v$. However, this contradicts that w_0 cannot be 0 for an element that cancels out $u^n v$.
Let us look at what such a cancellation appears to be in the spectral sequence. On the E_1 page, we see only $u^n v$ and w_0 (as every other w_i is not in the kernel of d'). These two terms must be canceled out, but the only induced differential that could cancel them out would have to be of degree n. Therefore, we have this differential appears on the E_{n+1} page, which implies that $u^n v$ appears on $n+1$ page of the spectral theorem, but not on the $n+2$ page.
We now want to show that this pairing data carries over to $BN^{bullet,\infty}$. However, this follows from the fact that w is the element of minimal filtered width that cancels $u^n v$, so $u^k w$ is the element of minimal filtered width that cancels out $u^{n+k} v$. □

Obviously, we would like the converse statement to also be true, and it is.

Lemma 6.8. *Let v and w be u degree 0, and suppose that $u^{n+k} v$ is cancelled out by $u^k w$ on the $n+1$ page of the spectral sequence. Then v is u torsion of degree n.*

Proof. It is clear that w can be no more than u^n torsion, because $u^k w = v$. Suppose that v had u torsion less than n. Then by the above lemma, it would be cancelled out in the spectral sequence before the $n+1$ page. This is a contradiction. □

From here, we have that the filtered theory and the $\mathbb{F}_2[u]$ theory are equivalent. We have a dictionary between these two theories:

- u torsion corresponds to the "width" of the pairing in the filtered theory.
- Non-homogenous u degree of generator corresponds to no being paired in the spectral sequence.

7. Where to look from here

Some thoughts on where to look to develop the theory further.

- We had that the map $k^* : BN^{\bullet,\bullet} \to BN^{\bullet,\bullet}$ was not an exact map– not even a differential! Can knowing just the quantum degrees of $BN^{\bullet,\bullet}$ and the reduced theory $\widetilde{BN^{\bullet,\bullet}}$ give additional structure on the theory besides the rate of convergence?
- Even better, does knowing the reduced $\mathbb{F}_2$ and full $\mathbb{F}_2$ theory give additional information on pairings of elements in the filtered $\mathbb{F}_2$ theory?
- We described Schumakovitch's s map, via a cutting relationship on cobordisms. Does this technique extended to other tautological functors given by Bar-Natan in his paper?
- Does the presence of an additional spectral sequence to compute $BN^{\bullet,\bullet}$ significantly improve computation time?

References

[Bar02] D. Bar-Natan. On Khovanov's categorification of the Jones polynomial. *ArXiv Mathematics e-prints*, January 2002.

[Bar04] D. Bar-Natan. Khovanov's homology for tangles and cobordisms. *ArXiv Mathematics e-prints*, October 2004.

[HW10] Matthew Hedden and Liam Watson. Does Khovanov homology detect the unknot? *Amer. J. Math.*, 132(5):1339–1345, 2010.

[Kho99] M. Khovanov. A categorification of the Jones polynomial. *ArXiv Mathematics e-prints*, August 1999.

[Kho02] M. Khovanov. Patterns in knot cohomology I. *ArXiv Mathematics e-prints*, January 2002.

[KM11] P. B. Kronheimer and T. S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.

[Lee02] E. S. Lee. The support of the Khovanov's invariants for alternating knots. *ArXiv Mathematics e-prints*, January 2002.

[Lee05] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197(2):554–586, 2005.

[MO05] C. Manolescu and B. Owens. A concordance invariant from the Floer homology of double branched covers. *ArXiv Mathematics e-prints*, August 2005.

[OS05] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[Ras05] Jacob Rasmussen. Knot polynomials and knot homologies. In *Geometry and topology of manifolds*, volume 47 of *Fields Inst. Commun.*, pages 261–280. Amer. Math. Soc., Providence, RI, 2005.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[Shu04] A. Shumakovitch. Torsion of the Khovanov homology. *ArXiv Mathematics e-prints*, May 2004.

[Tur04] P. Turner. Calculating Bar-Natan's characteristic two Khovanov homology. *ArXiv Mathematics e-prints*, November 2004.

[Tur06] P. Turner. Five Lectures on Khovanov Homology. *ArXiv Mathematics e-prints*, June 2006.