A Survey of Graph Cohomology and Perturbative Methods

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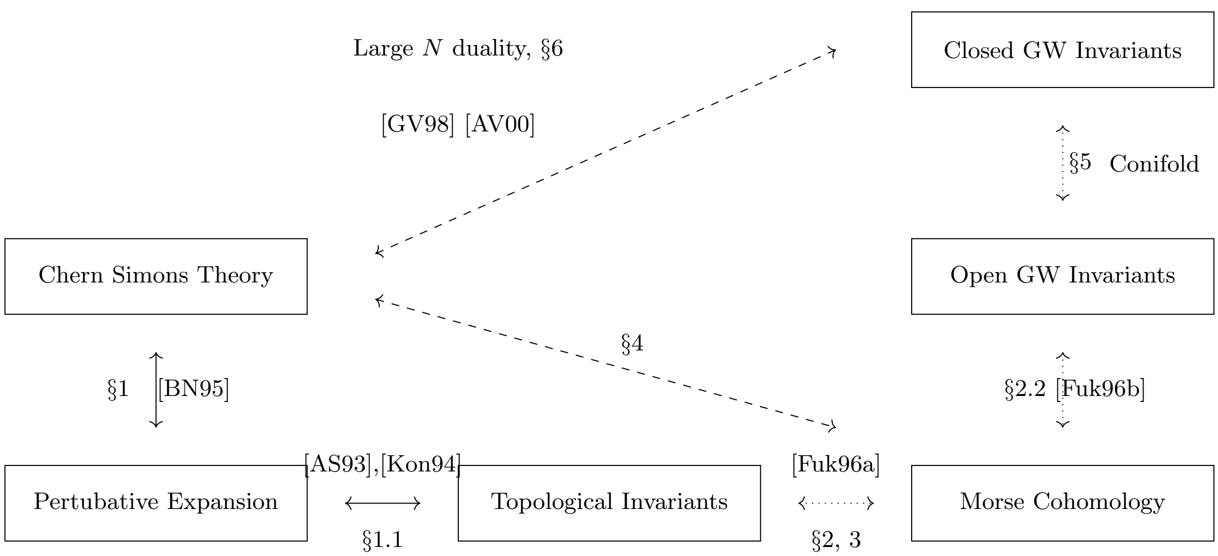
Abstract

This is a set of seminar notes which survey several different connections between Gromov-Witten invariants and Chern-Simons theory.
Using perturbation Chern-Simons theory, Bar-Natan, Kontsevich and others were able to construct a 3 manifold an invariant that captures the contributions of the second term of Witten's Chern-Simons expansion. This is sometimes called the "two-loop contribution" to Chern-Simons theory, and the resulting invariant can be computed without using perturbative techniques. in [**m**]imic these constructions using graph cohomology in Morse theory, and find that the Feynman diagrams that appear in our gauge-theoretic construction show up again in the Morse theory. By viewing Morse theory as Lagrangian Floer theory of the zero section, we'll see the conjectured "Large N-duality" relating the Chern-Simons potential function to Gromov-Witten invariants in the cotangent bundle. Our primary source for this is [Fuk96a].

Outline

Here is a guide to the topics which are outlined in these notes, as well as the relations between them:



1 Perturbative Chern-Simons Theory

I'm going to start with a brief exposition on Perturbative Chern Simons Theory [1]. Roughly speaking, the Chern-Simons theory associates to a bundle $G \to M$ a number, which is computed by the "path

[1] I currently do not have a comprehensive understanding of this, and would be happy for input on this!

integral" of the Chern-Simons functional. While we cannot do this computation mathematically; there are several different ways we can access this theory mathematically.
One way to access this theory is to use tools from TQFT. This was done by [RT91].
A second way to get a handle on this theory is to formalize the perturbative techniques used to compute the Chern-Simons path integral; this technique was used to produce well-defined mathematical invariants in [BN95]. This is the approach we'll be taking here.

Definition 1 (Chern Simons Theory). *Let $G \to M$ be a bundle, and A a connection on G. Then Chern-Simons functional on a connection*

$$cs(A) = \frac{1}{4\pi} \int_{M^3} \operatorname{tr}(3A \wedge dA + 2A \wedge A \wedge A).$$

The Chern Simons path integral associated to $G \to M$ is given by the path integral

$$\mathcal{W}(M^3, k) = \int_{\mathcal{A}} e^{ikcs(A)} \mathcal{D}A$$

On the nose, this integral is not defined. In the case where M is a rational homology sphere, one may hope to compute this integral by perturbative integral instead, as the Chern-Simons functional has critical values at flat connections and M will have finitely many such connections. Furthermore, since the Chern-Simons functional has a quadratic term with a cubic correction, such an expansion in terms of integrals associated to Feynman graphs will expand over cubic graphs.
To compute this expansion around the critical points of $cs(A)$, look at the asymptotics of $\mathcal{W}(M^3, k)$ as k goes to infinity. Here are the steps outlined in [Kon94] on how we can access invariants here.
If we believe that the numbers $W(M^3, k)$ exist, then there should be some understanding of the asymptotic behavior of these numbers as k goes to ∞. This integral should, as in the Morse case, localize around the critical points of $cs(A)$, which are the flat connection.
Our asymptotic expansion will be of the form

$$\sum_{A \text{ flat connection}} e^{cs(A)} \frac{k^{d_A}}{R(A)} \exp\left(\sum_{n \in \mathbb{N}} a_{n,A} k^{-n} \right)$$

where d_A and $R(A)$ are invariants related to the Reidemeister Torsion. Speculatively, one could hope that each coefficient $a_{n,A}$ gives us an invariant of the space. For the rest of this talk, we'll be concerned with understanding the term $a_{2,0}$.
Physical intuition tells us that this expansion can be reformulated by Feynman integral type computation, and the second order term is given by computing over 2 loop diagrams.

Remark 1. *I don't really know the Feynman-Diagram Calculus to detail the next few steps. Here is an amalgamation of ideas, taken from [BN95], [Kon94] and [AS93]. [Ekh13] is another quick introduction.*

- *By fixing a Riemannian metric on M, we have "fixed gauge," which gives us a propagator form (this is a 2-form P on M^2 with values in $\mathfrak{g} \otimes \mathfrak{g}$. This form is smooth and closed outside the diagonal $M \times M$, and its exterior derivative is equal to the delta-form on M_{diag}. For us, it is the Green form, satisfying*

  $$d\left(\int_y P(x, y) \wedge u(y) dy \right) = u(y)$$

  This Green form is suppose to tell us the contribution of the Feynman integral coming from each edge. Right now, this is a 2-form, but in general we want this to be a 2 form with values in $\operatorname{Hom}(E, E)$*, where E is the bundle that we are taking connections in.*

  Remark 2. *It is important to notice that this propagator term is morally some kind of inverse to the exterior derivative. We'll exploit this later when we try to deform this propagator.*

- *The Feynman diagram calculus tells us that the contribution from each graph Γ is*

  $$\int_{M^{2n} \setminus \Delta} \operatorname{tr} \prod_{i=1}^{3n} P(x_{l_i}, x_{r_i})$$

where x_{l_i}, x_{r_i} are the ends of the labels of the edge l_i, and tr *is given by tensor product over all vertices the skew-symmetric invariant 3-linear functionals on $\mathfrak{g}$,*

$$X_1 \otimes X_1 \otimes X_2 \mapsto \operatorname{tr}(X_1 X_2 X_3 - X_3 X_2 X_1)$$

The reason we can do this is that $(\mathfrak{g}^{\otimes 2})^{\otimes 3} = (\mathfrak{g}^{\otimes 3})^{\otimes 2n}$ which is made by the labeling of the vertices and edges.

- *In the 2 loop case, symmetries tell us that the only diagram we need to consider (for first order) is the Θ shaped graph, which gives contributions of the form*

Theorem 1. *To first order, the 2 loop contribution to the large k Chern-Simons theory is given by*

$$\int_{(x,y)\in M\times M} \operatorname{tr}_\Theta(P(x,y) \wedge P(x,y) \wedge P(x,y))$$

which, when viewed with corrections, is an invariant of the manifold.

Importantly, this claim can be proved using machinery from [BN95], or from [AS93], and is cited in [Kon94]. It is a mathematically defined invariant.

1.1 How to relate this to other invariants

We now have an invariant presented in some purely topological data. Let's try to massage this a little bit:

- Notice that if P was a form defined on M^2, that $dP = T_\Delta$, the delta distribution of the diagonal.
- There is one invariant that we can pull out of this possibly, which would be

$$\int_{M\times M} T_\Delta \wedge T_\Delta.$$

  Of course, we can't compute this, but it should (morally) be the intersection number of the diagonal, which is a topological invariant.
- We'll try to imitate this construction using Morse theory. Notice that if we take $M(f)$ to be the pairs of points in M which lie on the same flow-line of f, then one component of the boundary is the diagonal f. We expect that the intersection of $M(f) \cap M(f) \cap M(f)$ to compute something similar to $\int_{M^2} P \wedge P \wedge P$.

2 Graph Cohomology

We start by taking a short review of Witten's approach to Morse theory, which we'll return to towards the end of the talk.

2.1 Witten's Morse Theory

Witten's approach to Morse theory [Wit+82] was to relate the harmonic forms on M to it's Morse theory by perturbative methods. Recall that a form $\omega \in \Omega^k(M)$ is *harmonic* if

$$\Delta\omega = 0$$

where $\Delta = d + d^*$. The space of harmonic k-forms is denoted

$$\mathcal{H}^k(M) \simeq H^k(M, \mathbb{R})$$

and is isomorphic to the de-Rahm theory of M.
Witten first considers a version of the exterior derivative deformed by a Morse function,

$$d_t = e^{-ft} d e^{ft},$$

where f is a Morse function. Similarly, one can define a *deformed Hodge Laplacian,*

$$\Delta_t = d_t + d_t^*$$

which has the property that for small t, the spectrum of this operator is similar to that of Δ, and therefore computes the harmonic forms on M.
However, for large t, the operator Δ_t has eigenfunctions localized around the critical points of f. By taking an expansion

$$\Delta_t = d_t d_t^* + d_t^* d_t = dd^* + d^* d + t \operatorname{Hess} f + t^2 (df)^2$$

one see that around noncritical points of f, is has a critical the behavior of H_t as the $t \to \infty$ is similar to $t^2(df)^2$ –which is very large. Therefore, there are no elements in the kernel here.
When we are at a critical point p, the Hessian is nonsingular, and we can now try to compute the eigenvectors of Δ_t. A local computation shows that there is a single element of small eigenvalue, which is a "deformed harmonic form" sitting in degree k, where k is the Morse index of f at p. This turns out to be the sharpest bound we can get on the space of harmonic forms *locally*.
To get a better understanding of the harmonic forms globally, we can construct the *Morse Complex*.

- For each p, we have the space of p-forms minimizing energy at the critical points x_i of Morse index p. Suppose there is a flow line from $x \to y$, which differ by Morse index 1. A calculation shows that this provides a better bound on the energy associated to that critical value (namely, the flow line contributes $\exp(f(x) - f(y))$ to the energy of that form. The forms which are the 0-eigenvectors are those in the homology of this new differential.
- The way that we will package up this data is the Morse Complex, which as a vector space is

  $$C^k(M, f) := \bigoplus_{p \in \operatorname{Crit}_k(f)} \mathbb{R}$$

  with differential defined by a weighted count of flow lines going between critical points

  $$\langle \partial p, q \rangle = \chi \# \mathcal{M}_{|_q^p}(f)$$

  where the weight $\chi = e^{f(p) - f(q)}$
- At this point, I want to introduce a useful variation of this theory. If we equip our manifold M with a local system, we can also build up an invariant which weights each flow line with $e^{f(x)-f(y)} e^{\eta(x)-\eta(y)}$, which computes the homology with local coefficients in η.

2.2 Graph Cohomology and the Fukaya Category

We could generalize this theory to look instead at directed graphs instead of flow lines. In the simplest example, we look at flow trees instead of flow lines [Fuk96b].

Notation 1. *From here on out, we're going to be talking about a bunch of different moduli spaces of flows. When we write $\mathcal{M}_G(f_1, \ldots f_k)$ we're going to be looking at the moduli space of graphs G in M where the edges are given by flow lines of f_i with incidence conditions specified by the f_i.*

We can use this setup to generalize the Morse homology to an A_∞ category.

Definition 2. *Let M be a compact Riemannian manifold. The* Fukaya-Morse *category is the A_∞ category specified by the following pieces of data:*

- *The objects of this category are Morse functions f_i on M.*
- *For two objects f_1 and f_2, the morphism complex between them is defined by*

  $$\hom^k(f_1, f_2) := C^k(M, f_1 - f + 2)$$

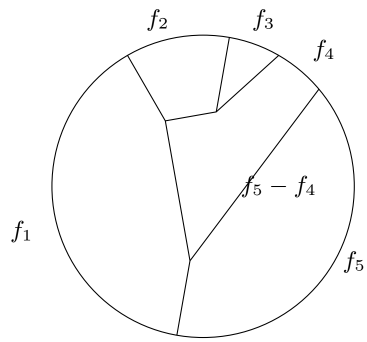
  which is equipped with the standard Morse differential.
- *The kth higher product*

  $$\mu_k : \hom(f_2, f_1) \otimes \hom(f_3, f_2) \otimes \cdots \hom(f_k, f_{k_1}) \to \hom(f_k, f_1)$$

  is defined by taking a count of the moduli space

  $$\mathcal{M}_Y(f_2 - f_1, \ldots, f_{k-1} - f_k, f_k - f_1)$$

  of ribbon trees Y with edges corresponding to the following flow lines.

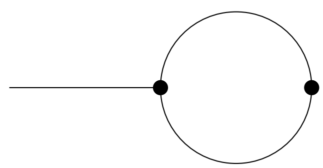


The A_∞ structure on this category comes from the fact that the space of these ribbon trees compactify to the Stasheff polyhedron.

Remark 3. *As an algebra, this is isomorphic to the full subcategory of the Fukaya category whose objects are given by hamiltonian deformations of the zero section in T^*M. We'll return to this at the end of the talk.*

2.2.1 Some Other Invariants

We could use other graphs. These give us elements of $\bigotimes H^\bullet(M) \otimes \bigotimes H_\bullet(M)$ based on the number of inputs and outputs we have. In particular



was studied in [BC94]. The dimension of this class is $\operatorname{ind}(a) - d$, where a is the index of the input critical point. This describes an element of $H^d(M)$, which is the Euler class [2]

3 Converting from Chern Simons to Graph Cohomology

From here, we follow [Fuk96a]. Recall that we desired to express the triple intersection of a submanifold which has, as its boundary, the diagonal in $M \times M$. At first glance, the following makes a good approximation to such a submanifold: we defined

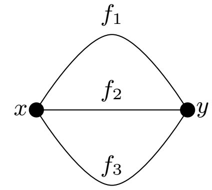
$$M(f) := \{(x, y) \in M \times M \mid \phi_f^t(x) = y\}$$

Then certainly one boundary component of this submanifold is the diagonal (the component that is associated to when $t = 0$.)
In order to make sense of an intersection of this manifold, we'll have to take a perturbation of it. So, we'll look at

$$M(f_1) \cap M(f_2) \cap M(f_3)$$

[2] One can see that it counts each Morse critical point with sign given by the index.

and count the points in this[3]. In graph cohomology, this corresponds to counting the flows of graphs that look like this:



Since this has no inputs and no outputs, it should give us a an element of $\mathbb{R}$ living in grading $3-d$. In the case where $d=3$, we get an actual number.
However, we have no reason to believe that a count of such graphs gives us any kind of invariant (as there is no output or input point to homology.) And in fact, the count of such graphs is dependent on the choice of functions f_i which we use to weigh the edges. In order to get an honest invariant, we'll have to throw in correction terms which account for us changing the functions f_i, work with local systems and a bunch of other things.

We'll also explore a version of this theory that counts by local systems instead. Let's write out a longer outline of this theory:

- To first order, we want to study the 3-edge graph. We'll take this to be the moduli space

  $$\mathcal{M}_\Theta(f_1, f_2, f_2) := \{(x, y; t_1, t_2, t_3 \in M^2 \times \mathbb{R}^3_+ \mid \phi^{t_i}_{f_i}(x) = y\}$$

  We'll call each element of this moduli space $I : \Theta \to M$.

- To each such I, and a local system η, we'll let the *weight of I by η* be

  $$\chi(I, \eta) := \sum_{\gamma_i} \mathrm{hol}_\eta(\gamma_i)$$

  where the γ_i are the four curves associated to different ribbon structures of I.

- We define our preliminary invariant

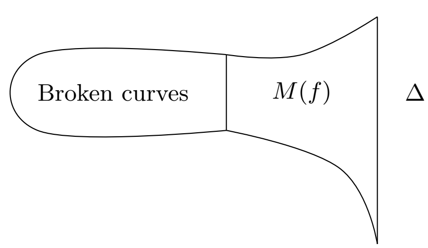
  $$Z^{prelim}_\Theta(f_1, f_2, f_3; \eta) := \sum_{I \in \mathcal{M}_\Theta} (f_1, f_2, f_3)\epsilon_I \chi(I, \eta)$$

  where I assume that ϵ_I is some kind of orientation count.

- In any case, this is suppose to be the Morse theoretic interpretation of the triple intersection of $M(f)$.

This gives us the first order theory. However, there are some correction terms that we need to throw in. There are 2 types of corrections that we'll need to account for.

1. At the start of the paper, we were searching for a space $\hat{M}(f) \subset M \times M$ so that $\partial \hat{M}(f) = \Delta$. However, $M(f)$ has another boundary component, given by the flow lines that go between two critical points.

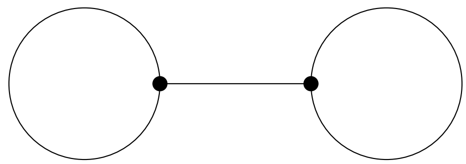
   

   We therefore want a space which has as its boundary the pairs of points critical points. When we glue these two spaces together, we'll have moved the undesired boundary to the interior of a large moduli space, and the count of objects in this moduli space will be the desired amount. We can

[3] Problematically, this is not a transverse intersection either, as all three submanifolds intersect the diagonal.

account for this type of correction by incorporating "broken diagrams."
Fukaya solves this by introducing a "combinatorial propagator" term which assigns a weight to each broken Θ diagram based on the holonomy of the diagram versus the holonomy of a preferred set of flows between critical points. Namely, when computing homology with local systems so that $H^\bullet(M, \eta) = 0$, there is an element of degree 1 in $\mathrm{End}(C^\bullet(M, \eta)$ so that $\partial\eta = \mathrm{id}$. We call this the combinatorial propagator. The correction that one needs to account for breaking of these Θ diagrams corresponds to diagram broken along this combinatorial propagator.

2. There is a second kind of correction that we need to consider, which is given by the difficulty of maintaining transversality at the diagonal. Notice that if f_i and f_j are two Morse functions, that $M(f_i)$ and $M(f_j)$ are not transverse at the diagonal, so we are not able to actually take their intersection properly. This can be viewed as what occurs when one of the edges in Θ goes to length zero. This can be viewed as a contribution coming from the graph Λ.



In fact, if we take the graphs up to symmetry, the defined invariant of Fukaya has

$$3(\text{Terms from } \Theta) + 2(\text{Terms from } \Lambda)$$

which looks a lot like the terms in the Chern-Simons functional.

Theorem 2. *Given 2 local systems, η_a and η_b, the quantity*

$$Z(f_1, f_2, f_3, \eta_a) - Z(f_1, f_2, f_3, \eta_b)$$

is independent of f_i and other choices made.

4 Conjectured Relation

In [AS93] and [Kon94], the leading term of the proposed 2-loop invariant is given by

$$\int_{(x,y)\in M^2} \mathrm{tr}_\Theta(P(x,y) \wedge P(x,y) \wedge P(x,y))$$

where P is a propagator term. Here, we'll pick a specific propagator term. Our goal will be to suitable deform this propagator term to suggest an equivalence between our Morse-Homotopy invariant and the 2-loop Chern-Simons invariant.
The propagator is morally an inverse to $d + d^*$ [BN95]. Let $G(t; x, y) \in Hom(\zeta_x, \zeta_y) \otimes (\bigoplus_i \Lambda^i_x \otimes \Lambda^i y)$ be a Green kernel of the Laplace operator, so that whenever $u : M \to \mathbb{R}$ is a function, letting

$$u(t, x) = \int_y \langle G(t; x, y), u(y)\rangle dV$$

has

$$\lim_{t\to 0} u(t, x) = u(x)$$
$$\frac{du}{dt} = -\Delta u$$

Then letting

$$P = (\delta \otimes *) \int_0^\infty G(t; x, y) dt$$

is a propagator. One idea is to take the approach that we did to Morse theory and compute this by adding in a deformation term to the Laplacian L. Modifying (now the metric) to be

$$\langle u, v\rangle_{f_i,\epsilon} = \int_M \langle (u(x), v(x)e^{f_i(x)/\epsilon} dS$$

We have that

$$\Delta^{\epsilon}_{f_i} = \Delta + \frac{1}{\epsilon} L_{\epsilon \,\mathrm{grad}\, f_i}.$$

where $L_{\epsilon \,\mathrm{grad}\, f_i}$ is the Lie derivative in the direction of the gradient. Similarly, let us define

$$P_{\epsilon, f_i} = (\delta_{\epsilon, f_i} \otimes *) G_{\epsilon f_i}(t; x, y)$$

and the invariant

$$\int_{M^2} \int_{t_1, t_2, t_2} Tr_{\Theta}(P_{\epsilon, f_1}(t_1; x, y) \wedge P_{\epsilon, f_2}(t_1; x, y) \wedge P_{\epsilon, f_3}(t_1; x, y)))$$

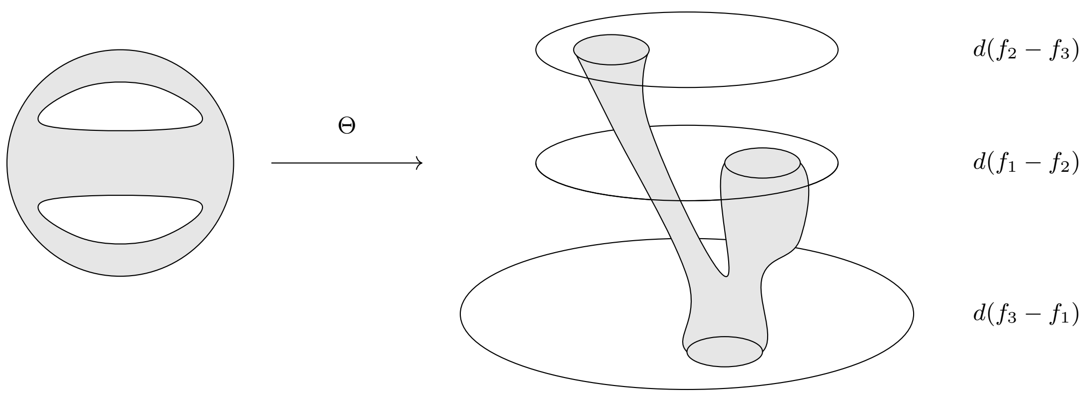
The limit as $\epsilon \to 0$ of this makes $G_{\epsilon, f_i}(t; x, y)$ supported on $M(f_i)$, so this integral gives us the leading term of our defined invariant. However, it is still unknown how to incorporate the correction terms.

5 Open String Theory and Conifolds

Here we talk about a speculative relation between the relation between Chern Simons theory, Morse theory, and our analysis.
One possible way to upgrade our Morse theory is to look at the Fukaya category.
We've already seen how we can put an A_∞ structure on the Morse homology by counting tree-like graph cohomology. The correspondence between the holomorphic disks with marked boundary points and the Morse flow trees is that we "thicken" up the graphs that we are counting into holomorphic disks.

Theorem 3. *The Morse A_∞ category is isomorphic to copies of the zero section in the Fukaya category.*

If we try to repeat this philosophy with the graph cohomology we have set up now, we should be examining thickenings of the Θ shaped ribbon graph. The count of this graph should correspond to an open GW-invariant with 3 boundary punctures, each mapped to different perturbed copies of the zero section.



Remark 4. *Several problems emerge.*

- *The first problem is that there isn't a unique way to extend the Θ graph to a ribbon graph. We have several different choices in this extension.*
- *The second problem is that the Fukaya category version works on the* differences *of Morse functions, not on the Morse functions themselves. This means that the Θ invariant that we are counting corresponds to the functions*

  $$\mathcal{M}_{\Theta}(f_1 - f_2, f_2 - f_2, f_3 - f_1)$$

  Unfortunately, this does not encompass a wide enough set of functions to capture the invariant that we want to construct. So, there is a discrepancy here.

If we could solve these two problems, we would have a firm connection between the perturbative terms of the Chern-Simons invariants and Gromov-Witten invariants. This has been predicted in a few places.

5.1 A bit of Background into Toric Geometry

This is a very sped-up introduction to Toric Varieties. A good reference to the algebraic geometry behind toric varieties is [CLS11]. I personally like [Clo09] a lot, which treats toric varieties from the viewpoint of differential geometry. Also, [Mar05] provides another viewpoint on locally constructing toric Calabi-Yau 3-folds from the perspective of physics.

Definition 3 (Toric Variety)**.** *An affine variety M is an* affine toric variety *if there exists an open dense $(\mathbb{C}^*)^n \subset X$ so that the natural action of $(\mathbb{C}^*)^n$ on itself extends to a holomorphic action on X*

We call $(\mathbb{C}^*)^n$ the *big torus*, and its action on X gives us a combinatorial characterization of a toric variety.

Definition 4 (Character)**.** *Let T^n be an algebraic torus. A* character *of T^n is a holomorphic map $\chi : T^n \to \mathbb{C}^*$ which is a homomorphism of toric action.*

Characters form an abelian semi-group, with operation

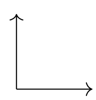
$$(\chi_u + \chi_v)(x) = \chi_u(x) \cdot \chi_v(x).$$

Claim 1. *The set of characters for $(\mathbb{C}^*)^n$ is $M = \mathbb{Z}^n$.*

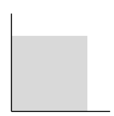
Definition 5. *A* cocharacter *of the torus is a map $\lambda_u : (\mathbb{C}^*) \to T^n$ which is a homomorphism of the toric action.*

The character and cocharacter lattice have a natural pairing which records the degree of the composition.
Given a toric variety X, we ask "which characters on $(\mathbb{C}^*)^n \to \mathbb{C}^*$ extend to maps $X \to \mathbb{C}^n$. ? This gives an affine semigroup of the characters. The data of this affine semigroup determines the toric variety, and we can combinatorially encode this data in the form of a toric fan.
Symplectic geometers frequently use the dual data, which that of the moment map.

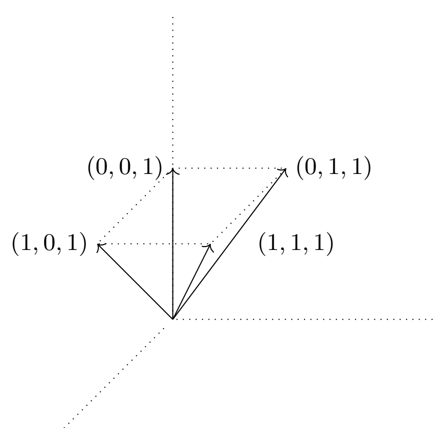
Example 1 (Complex Space)**.** $\mathbb{C}^n$ *is a toric variety. The action of $\mathbb{C}^*n$ on $\mathbb{C}^n$ is the usual one. Notice that the extendible characters are $\mathbb{N}^n$. The toric fan for $\mathbb{C}^n$ is given by*



The moment map (in the case of $\mathbb{C}^2$) is the first quadrant of $\mathbb{R}^2$.

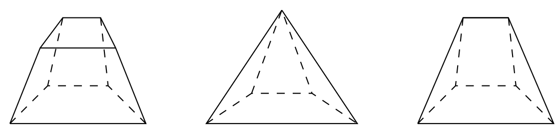


Example 2 (Conifold)**.** *As an affine variety, this is given by $xy - uv = 0$. Notice that this is not a smooth variety, as there is a rather bad singularity at the origin. The action is $(z_1, z_2, z_3) \mapsto (z_1, z_2, z_3, z_3^{-1} z_1 z_3)$. The characters that extend are $(0,0,1), (1,0,1), (0,1,1)$ and $(1,1,1)$, giving us the following fan:*



The non-smoothness at the origin comes from the fact that vertex at the origin has vectors leading to it which do not form a basis.
There are several ways to smooth out this singularity

- *One is to look at the blow up at the singularity. This is a pretty violent operation, and it involves adding in an additional ray to our fan, and making the 1 cone into 4 cones.*
- *The other operation is called the* small blow-up*, which involves adding in $\mathbb{CP}^1$ along divisor, splitting our 1 cone into 2 cones. The pictures are easier to draw on the moment map side:*



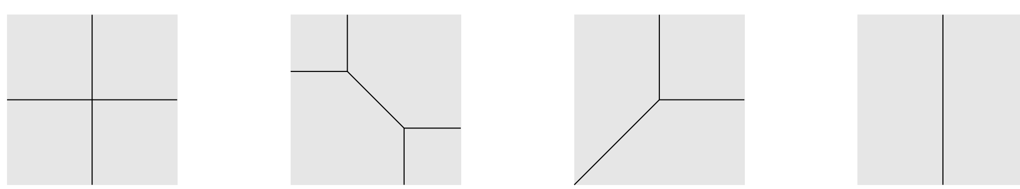
Blow Up *Conifold* *Small Resolution*

Both of these are examples of non-compact toric manifolds. In fact, the resolved conifold, conifold and $\mathbb{C}^n$ all give us good examples of *toric Calabi-Yaus.*
On any toric-Calabi Yau, we have a dimensional reduction of both the toric fan and moment polytope diagrams given by a choice of trivialization of the canonical bundle:

- One can prove that the anticanonical divisor is the sum of the toric divisors.
- If a toric variety is Calabi-Yau, there is a function whose zero is the union of the toric divisor with multiplicity one. This gives us a map from $M \to \mathbb{C}$ which respects the toric fibration. This map can be seen as a character by taking a vector $v \in M$ so that $\langle v, D \rangle = 1$ for every element in the toric fan.
- This means that we have an associated projection to the orthogonal plane to v, and we can instead draw the associated moment map and fan in the this orthogonal projection.

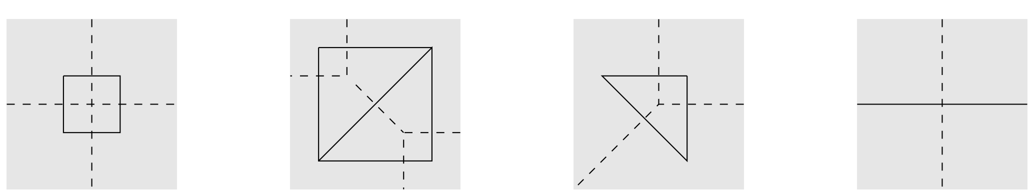
Definition 6. *The associated orthogonal graph of the moment map for a smooth toric Calabi Yau is trivalent and is called the pq-*web *of M.*

Example 3. *Here are a few pq webs. Each open edge represents a $\mathbb{C}^*$, each half open edge a $\mathbb{C}$ and each closed edge a $\mathbb{CP}^1$.*



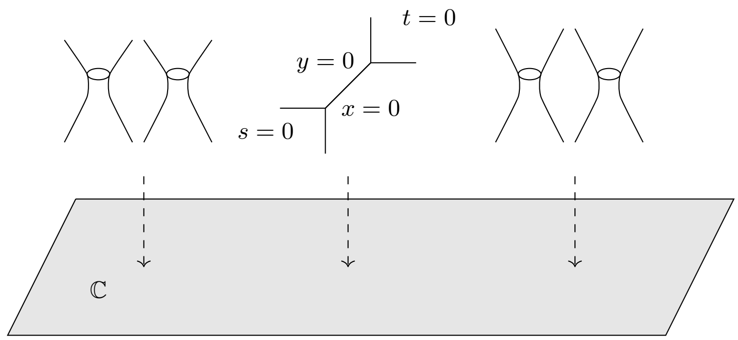
Conifold $O(-1) \oplus O(-1)$ $\mathbb{C}^3$ $\mathbb{C}^2 \times \mathbb{C}^*$

Notice that each trivalent vertex is represents a local toric $\mathbb{C}^3$ coordinate chart. By changing the lengths of the edges in these graphs we modify the Kähler form on the space, probing the symplectic structure. This also gives us a quick and easy way to compute $H^2(M)$, as it is equal to the number of closed edges in this graph.
The toric fans can be seen as the dual graph *of these pq-webs.*



Conifold $O(-1) \oplus O(-1)$ $\mathbb{C}^3$ $\mathbb{C}^2 \times \mathbb{C}^*$

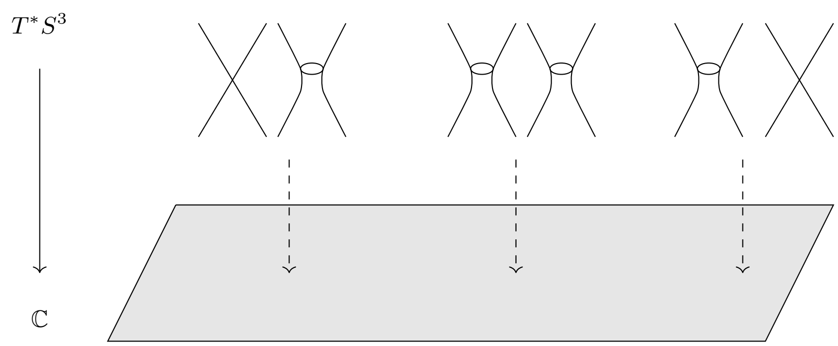
The function $\chi_v : M \to \mathbb{C}$ gives us a Lefschetz-Bott fibration of our space, with general fiber $(\mathbb{C}^*)^2$ and a singular fiber at the origin with degenerate locus determined by the *pq* web. For example, the resolved conifold gives us the following picture of a Lefschetz fibration



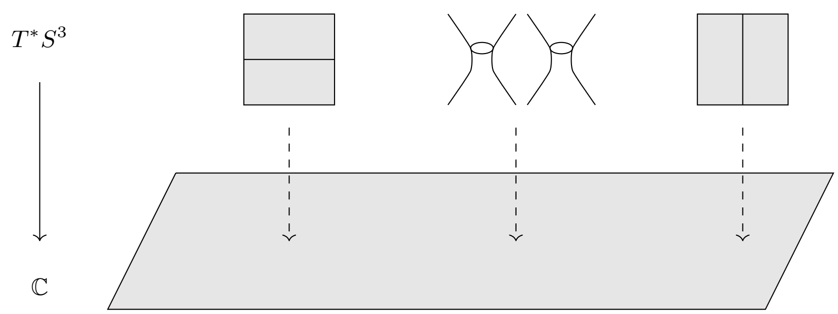
5.2 Inserting Singularities along different fibers

There is nothing special about us having the singularity in this fibration above zero. In practice, we could modify many of these generic $\mathbb{C}^* \times \mathbb{C}*$ fibers to be singular toric fibers instead.

Example 4. *For example, let's try replacing 2 fibers with singular fibers : a $\mathbb{C}^* \times \mathbb{C}$ in one fiber, and a $\mathbb{C} \times \mathbb{C}^*$ in at a different spot:*



With the pq web picture, we might draw this:



This happens to be a Lefschetz fibration of T^*S^3, equipped with the standard symplectic structure.

There is now a clear deformation that we can make of these manifolds: we can take two of the singular fibers and "mash" them together by bringing them closer to each other. This is a deformation of our space, and we'll get a new toric variety.

Claim 2. *The pq web of the resulting toric manifold is the disjoint union of the pq web associated to each of the singular fibers. In fan language, the new fan projection is the Minkowski sum of the previous two fans.*

This type of deformation (called a Gorenstein resolution of a toric singularity) was studied in detail by [Alt97], and the Lefschetz fibration viewpoint was observed by [Gro00] to give an SYZ fibration. These resulting fibrations have been studied by [Lau14] to construct SYZ mirrors to these spaces.

5.3 Conifold Transition

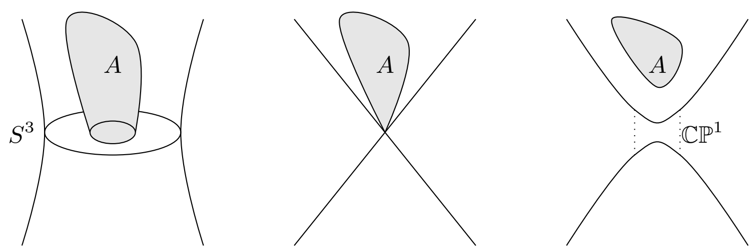
Notice now that for the singular conifold, we have 2 methods to produce a smooth Calabi-Yau 3 fold:

1. We can take the *small resolution* or *crepant resolution* of the conifold to the bundle $O(-1) \oplus O(-1) \to \mathbb{CP}^1$. We have some choice here when determining the symplectic structure, namely the area of the resulting added in $\mathbb{CP}^1$.
2. We can view this as a Gorenstein resolution of a toric singularity, and smooth this out to a T^*S^3. Here, we have a different choice, which is determining the volume of the sphere we add in.

This operation is called a conifold transition, and it generalizes to other Gorenstein resolutions.

Conjecture 1. *The Open Gromov-Witten theory of T^*S^3 with punctures landing on disjoint copies of S^3 is the same as the closed Gromov Witten Theory of $O(-1) \oplus O(-1)$.*

The motivating picture of this conjecture is the following:



6 Large N duality

We now have all of the pieces to relate Gromov-Witten theory with Chern Simons theory.

Conjecture 2 (Large N duality Principle)**.** *There is a change of coordinates relating the Chern-Simons potential for S^3 and the Gromov-Witten potential for $O(-1) \oplus O(-1)$.*

Here is some history on the conjecture:

- These predictions come from physics originally. 't Hooft made the first predictions that gauge theories could be expressed in the large 1/N expansions to be string theory [Hoo73]. In this physics literature, this appears to be first found by Maldacena in ADS/CFT correspondence [Mal99].
- The version that we are working with was predicted by Gopakumar-Vafa, and is either called Gopakumar-Vafa duality or Large N duality [GV98]. This physical prediction related the following three theories:
  - The Large N expansions of Chern Simons Theory on a sphere.
  - The open Gromov Witten invariants of T^*S^3 with boundary punctures wrapped around different perturbations of the zero section.
  - The closed Gromov-Witten invariants on the *conifold transition* of the zero section.

  A great reference for this relation is [AK06], which thoroughly and rigorously proves Large N duality. The Chern-Simons invariants and GW-invariants are computed for both S^3 and $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ respectively, and shown to be the same up to a correction term of

$$5/12 \ln x + \zeta(3)x^{-2} - \frac{1}{2}\ln(2\pi) - \zeta'(-1)$$

Recently, this tool has been used to make conjectures about the open Gromov-Witten potential related to another Lagrangian. This has been primarily studied by Aganagic and Vafa who used it to develop their "Q-deformed A-polynomial.
In Chern-Simons theory, one can modify the theory to incorporate knots by changing to potential function with a holonomy term coming from a knot. This modification is called "incorporating a Wilson loop," and the associated invariants are related to the Jones polynomial of the knot.

6.0.1 Application I: Knot Theory

More recently, the work of Aganagic and Vafa tell us how to take into account Wilson loops in Chern-Simons theory into this invariant [AV00]. In this setting, the Chern Simon theory is associated to the open disk potential of the Lagrangian conormal to a knot. Depending on whether you work in T^*S^3 or the conifold resolution of the knot, there is a term associated to the symplectic area of the $\mathbb{CP}^1$ factor thrown in. Again, as these are invariants that we are just starting to be able to compute, we can show that this conjecture holds with some correction terms. A good overview text of this is [Mar05].
Conjecturally, these open Gromov-Witten invariants match the Knot contact homology invariants of [Ng05], which also compute a type of Open Gromov-Witten invariant from the unit conormal bundle of a knot. This result has been check computationally in many cases in [AENV+14]

6.0.2 Application II: Open GW Invariants

Another application (by looking at the same kind of Lagrangians) is computation methods for open Gromov-Witten in a Toric Calabi-Yau 3-fold. Gromov-Witten invariants are notoriously difficult to compute, and the Chern Simons theory is comparatively easier. One of the goals of [AKMV05] and [LLLZ09] is to compute the open Gromov Witten potential of Toric manifolds by splitting them into small sections that look like resolved conifolds, and gluing together these conifolds. At each conifold, one computes the relative open Gromov Witten invariants by using Large N duality and doing the computation on the Chern-Simons side.

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