Some Equivalent Theorems in Extremal Combinatorics

Jeff Hicks

April 7, 2012

What is Extremal Combinatorics

Extremal Combinatorics is the study of the maximum or minimal behaviors in combinatorial objects

Problems!

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- Marraiges

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- Plumbing
- Marraiges
- Hard Choices

A Problem with the Pipes

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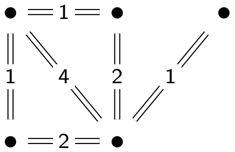
- How much water can flow from the source to the target?
- Which pipes do you have to cut in order to prevent water from flowing?



Networks for Beginners

Definition

A **network** *is a collection V of* **vertices** *and a subset $P \subset V \times V$ of* **pipes**. *We have a function $C : P \to \mathbb{N}$ called the* **capacity** *function.*

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$$\begin{tikzcd}
\bullet \arrow[r, "1"] \arrow[d, "1"] & \bullet & \bullet \arrow[dl, "1"] \\
\bullet \arrow[r, "2"] & \bullet \arrow[ul, "4"] \arrow[u, "2"] &
\end{tikzcd}$$

Figure: Network= Directed graph with Weighted Edges

Definition

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- *The flow along a pipe is less than the capacity of the pipe*
- *Total water is conserved, except at the source and target.*

Networks for Beginners: Flows

$$
\begin{tikzcd}
\textit{Source} \arrow[r, "1" description] \arrow[d, "2" description] & \bullet \arrow[d, "2" description] & \bullet \arrow[dl, "3" description] \\
\bullet \arrow[r, "2" description] & \textit{Target} \arrow[ul, "4" description] &
\end{tikzcd}
$$

Figure:

We say that the **flow between Source and Target** is the sum of the flows leaving Source or the sum of the flows entering Target.

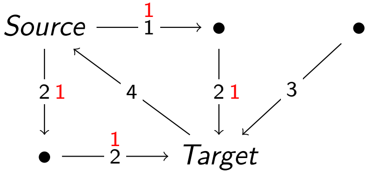

Figure: A possible flow in red.

We say that the **flow between Source and Target** is the sum of the flows leaving Source or the sum of the flows entering Target.In the above example, the flow between Source and Target is 2.

Maximal Flow

A flow is called maximal if it is the largest possible flow.

$$\begin{tikzcd}
\mathit{Source} \arrow[r, "1"] \arrow[d, "2"] & \bullet \arrow[d, "2"] & \bullet \arrow[dl, "3"] \\
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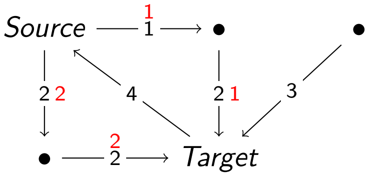

Figure: A Maximal Flow of 3

Networks for Beginners: Cuts

Definition

If the removal of a set of pipes H means that there is no flow from s to t , we call H a **cut**. *The* **capacity of the cut** *H is the sum of the capacity of the pipes in the cut.*

$$\begin{tikzcd}
Source \arrow[d, "1"] & \bullet \arrow[l, "1"] & \bullet \arrow[dl, "1"] \\
\bullet \arrow[r, "2"] & Sink \arrow[u, "2"] \arrow[ul, "4"] &
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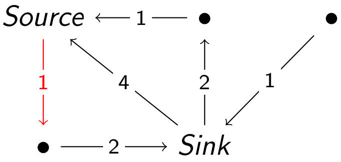

Figure: A cut of s and t of capacity 1.

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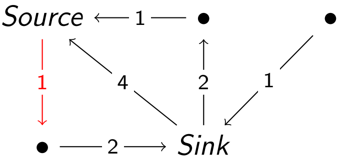

Figure: A cut of s and t of capacity 1.

A cut is called minimal if it has the smallest possible capacity.

The Max-flow Min-cut Theorem

Theorem (Max-flow Min-cut)

Maximum flow $=$ *Minimum Cut*

The Max Max-flow Min-cut Theorem: An Example

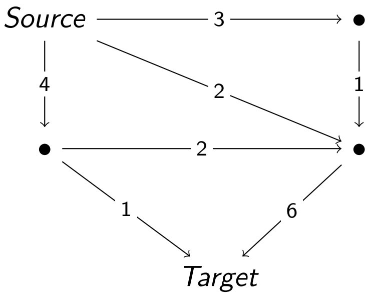

Figure: A network

The Max Max-flow Min-cut Theorem: An Example

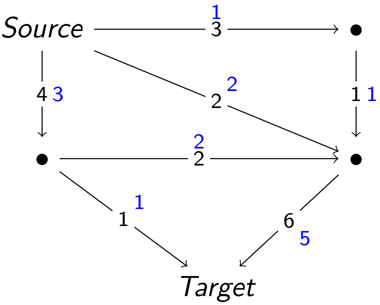

Figure: A network a flow of 6

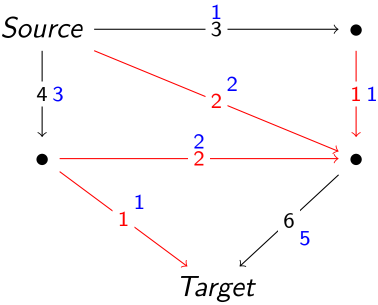

Figure: A network a flow of 6 a cut of 6

Max Flow Min Cut (MFMC) is a very powerful Theorem. Let's look at a seemingly unrelated problem.

The basic problem is as follows:

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- You are a matchmaker who happens to be a mathematician
- They boys and girls are not so picky, and have given you lists of who they are willing to marry
- How many happy couples can you make?

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Lets make this problem a little more formal.

Definition

A **graph** *is a collection* V *of* **vertices** *and a subset* $E \subset V \times V$ *of* **edges**.

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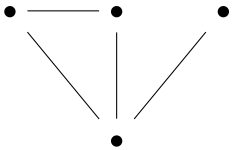

Figure: An example of a graph with 4 vertices and 4 edges

Definition

A set of vertices is called an **edge cover** *if every edge in the graph touches a vertex in the set.*

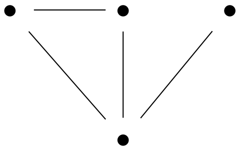

Figure: An example of a edge cover

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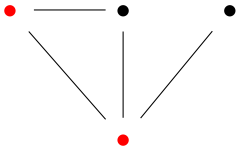

Figure: An example of a edge cover

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A set of edges is called an **matching** *if every edge is disjoint.*

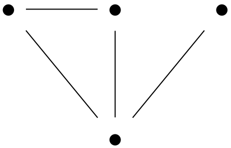

Figure: An example of a matching

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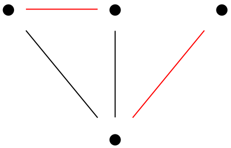

Figure: An example of a matching

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A graph is called **bipartite** *it has two parts, and the only edges are those which connect the two parts*

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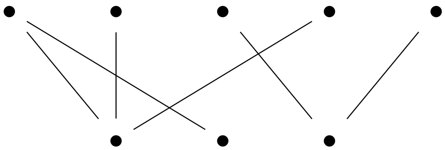
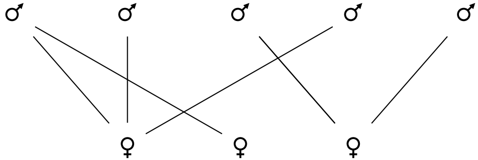


Figure: An example of a bipartite graph

Back To Marriages

We can make a graph out of the Marriage Problem



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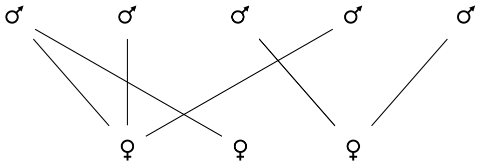

Vertices = People

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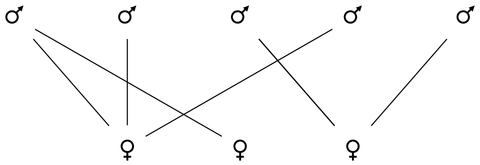

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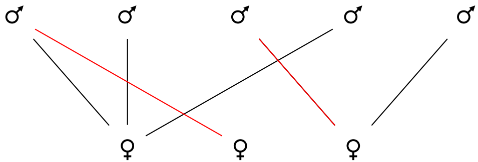

Vertices = People
Edges = Both people are happy if married
Possible sets of Marriages = Matchings

König's Theorem

Theorem

The size of a maximum matching in the Marriage Problem is equal to the size of a minimal edge cover

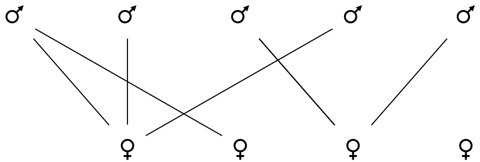

Figure: An example of a bipartite graph

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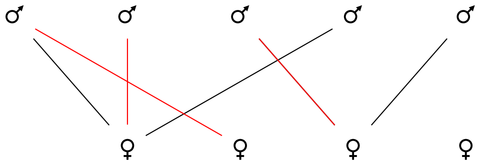

Figure: An example of a bipartite graph and a maximum matching

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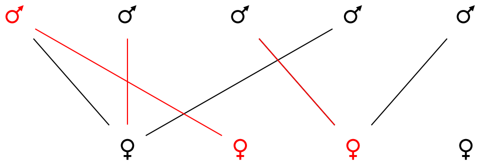
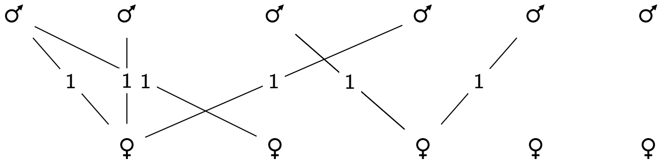


Figure: An example of a bipartite graph and a maximum matching and a minimal edge cover

The Max Max-flow Min-cut Theorem ⇒ König's Theorem



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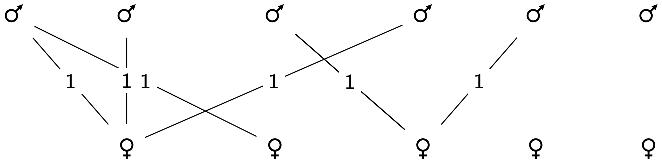

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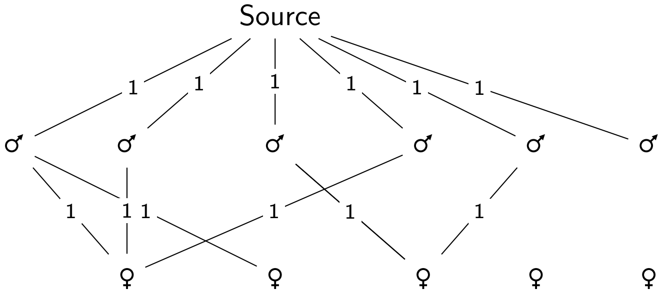

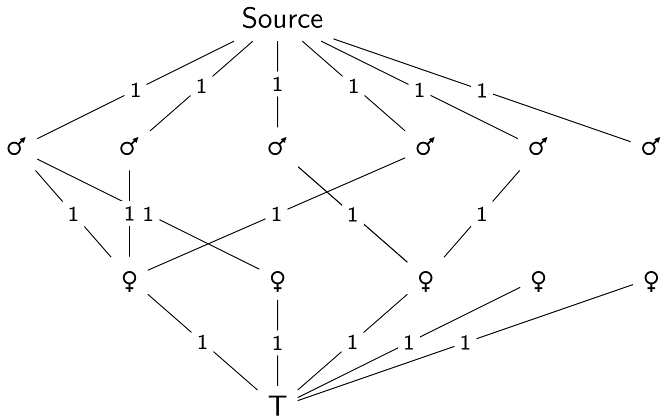
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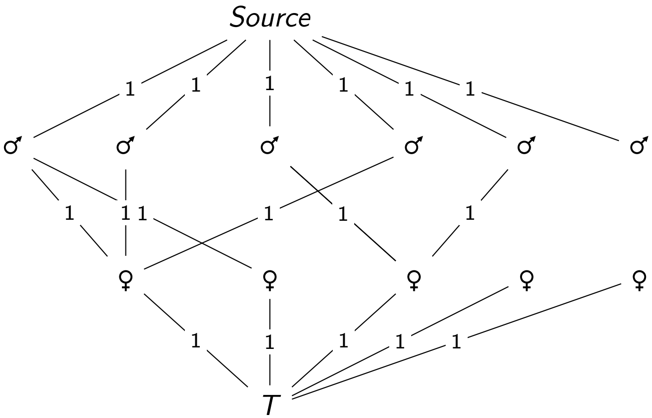
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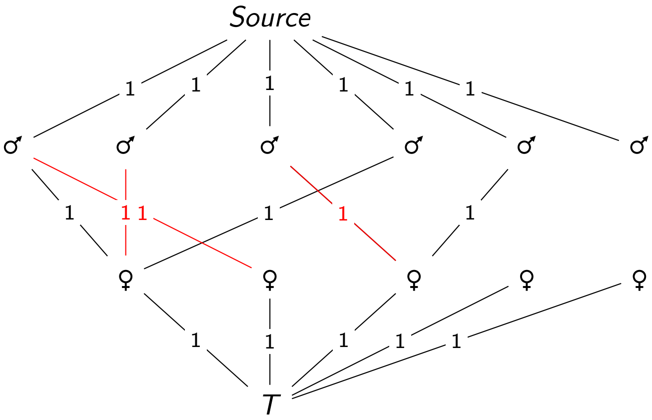
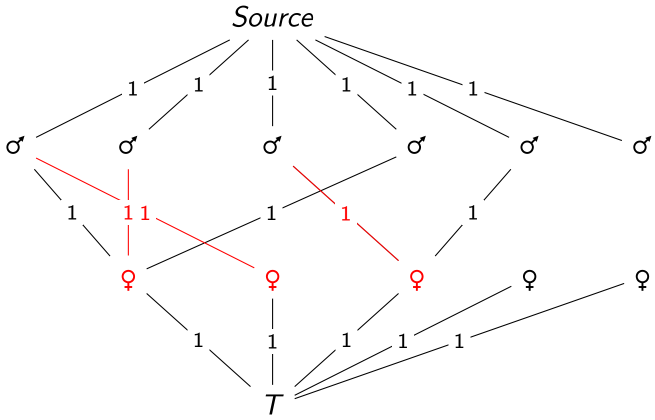


Max Flow = Max Matching



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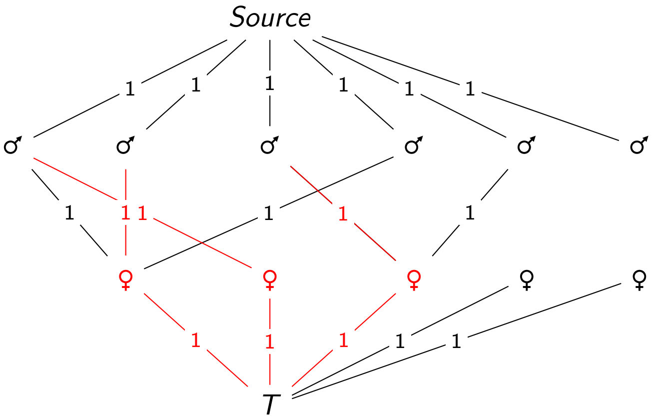

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Lets look at an unrelated problem, and see how König's theorem can help us

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- Largest Set of Incomparable Candy

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- How many piles of candy bars do I need to sort my candy so that each set is completely ordered

Candy and Hard Choices

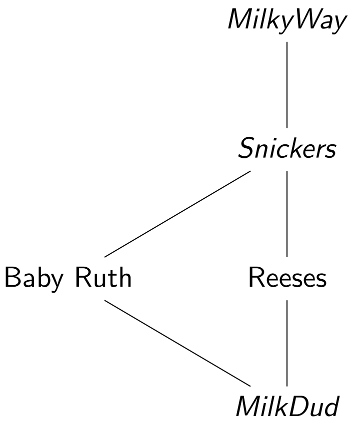
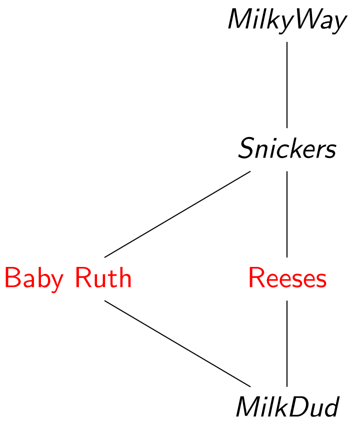


Figure: "Incomparable Items"

Candy and Hard Choices



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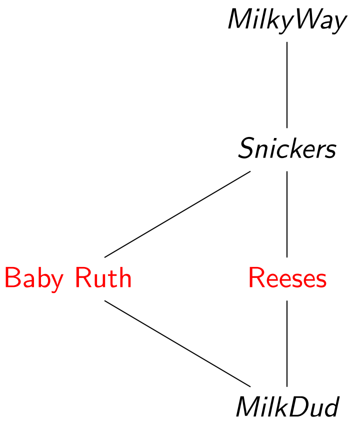

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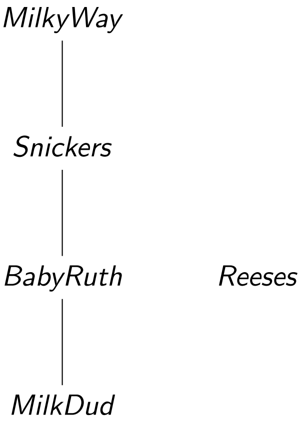

Figure: Each Pile is completely Ordered

A First Look at Posets

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- *(Reflexivity)* $a \leq a$
- *(Symmetry) If* $a \leq b$ *and* $b \leq a$ *then* $a = b$
- *(Transitivity) If* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$.

Warning! It is not necessarily the case that $a \leq b$ *or* $b \leq a$. *The items may not be comparable.*

A First Look at Posets

Example

A few examples

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- $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ with the usual ordering are all posets.

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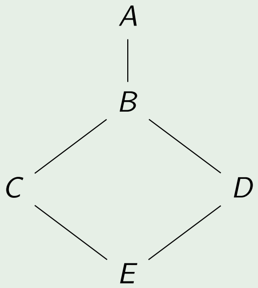

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Definition

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Definition

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Definition

A collection of chains $\mathcal{C} = C_1, C_2, C_3 \ldots$ is called a **chain covering of** *P if every element of P is contained in $\mathcal{C}$.*

Dilworth's Theorem

Theorem

Smallest Chain Cover = Max Antichain

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This theorem answers our question on candy bars:

Dilworth's Theorem

Theorem

Smallest Chain Cover = Max Antichain

This theorem answers our question on candy bars:
Smallest Set of Piles = Smallest Chain Cover

Theorem

Smallest Chain Cover = Max Antichain

This theorem answers our question on candy bars:
Smallest Set of Piles = Smallest Chain Cover
Maximum number of incomparable Candy Bars = Max Antichain

Candy and Hard Choices

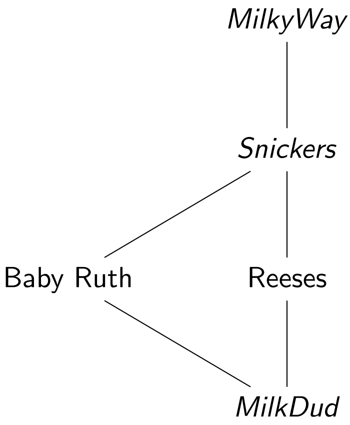
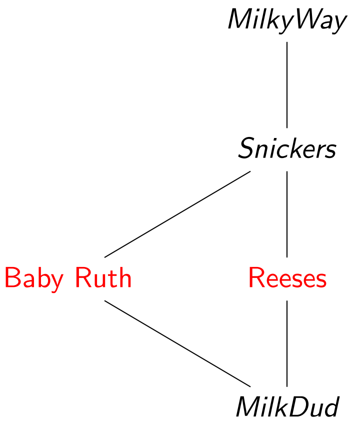


Figure: "Incomparable Items"



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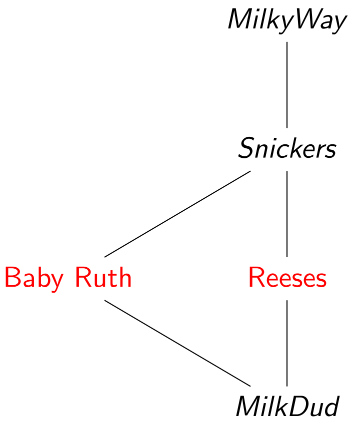
Candy and Hard Choices



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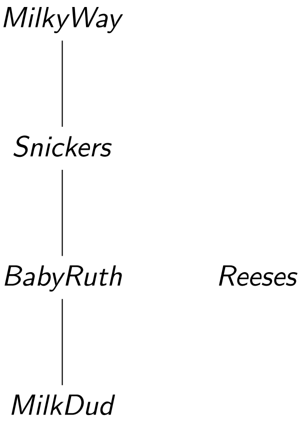
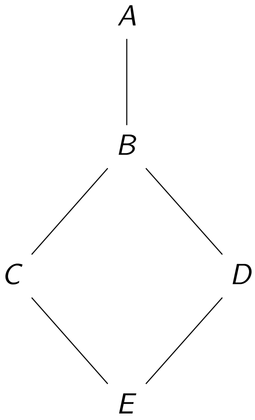


Figure: Each Pile is completely Ordered

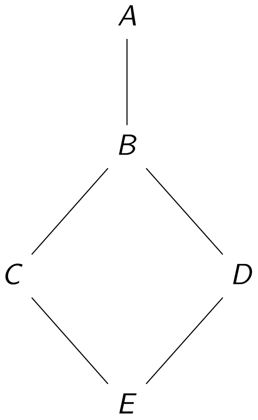
Proof of Dilworth's Theorem By König's Theorem

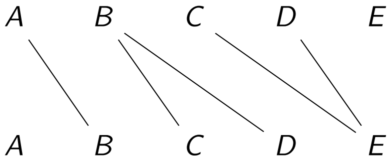
We create a Bipartite graph.
Vertices = Objects in Poset
Edge between a, b if $a < gb$



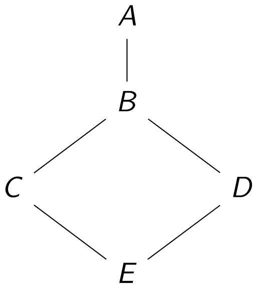
Proof of Dilworth's Theorem By König's Theorem

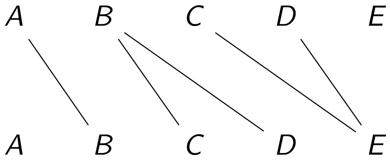
We create a Bipartite graph.
Vertices = Objects in Poset
Edge between a, b if $a <gb$





Let p be the number of elements in the Poset, and m the max matching/min edge cover





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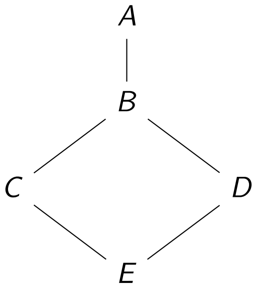

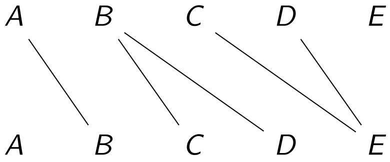

- If 2 vertecies are not in an edge cover, then there cannot be an edge between them.
  Max Antichain = Vertices not in Edge cover$=p-m$

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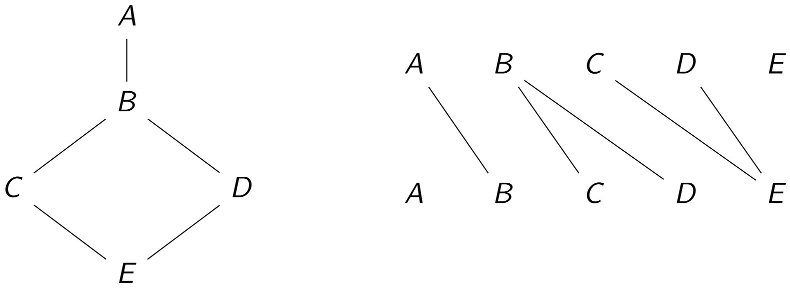

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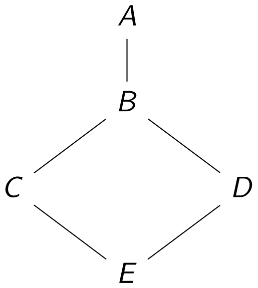

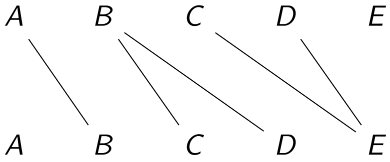

- If 2 vertecies are not in an edge cover, then there cannot be an edge between them.
  Max Antichain = Vertices not in Edge cover=$p - m$
- If a vertex is not bordering a maximum matching, then it is the "top" of a chain
  Min Chain Cover = vertices not bordering max matching=$p - m$

Other Equivelences

$$\begin{tikzcd} \text{König's} \arrow[rr] & & \text{Dilworths} \\ & \text{Maxflow-Mincut} \arrow[ul] & \end{tikzcd}$$

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There are actually many more theorems equivelent to these three

König's Theorem	Matchings on Bipartite Graphs

Other Equivelences

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Königs Matrix Theorem	Matrix decompositions