UC Berkeley
Department of Mathematics
Math 32– Final Exam Jeff Hicks

Location: LeConte 3

Date and Time: May 15, 2019; 7PM - 10 PM

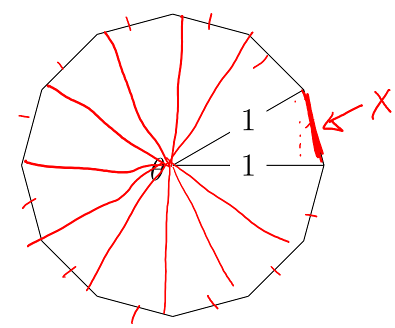
Name:

UID:

- Please do not turn over this page until instructed to do so.
- Please bring your student ID and have it available during the exam.
- This exam contains 14 problems, of which we will score 12 problems. Indicate to us which 12 problems you would like us to grade by checking the small box at the top of the page. We will only grade 12 problems. Each problem is worth 10 points, for a total score of 120 points on this exam.
- There are no notes or calculators allowed during the examination.
- Should you finish during the last 15 minutes of the exam period, please remain seated *until we have collected all of the exams* as other students will still be working.
- Solutions without work shown may not receive full credit. Box the solution you would like us to grade on each problem.
- Should you need extra space for work, you may use the empty pages included at the end of the exam. Mark clearly that your work is contained on a separate page in the original problem sheet.
- This exam contains 15 pages (including this page.)

1. Drawn below is a regular 12-gon.

all sides have same length



(a) (2 points) What is the measure of the angle θ in radians?

12 equal triangles making up 12-gon.

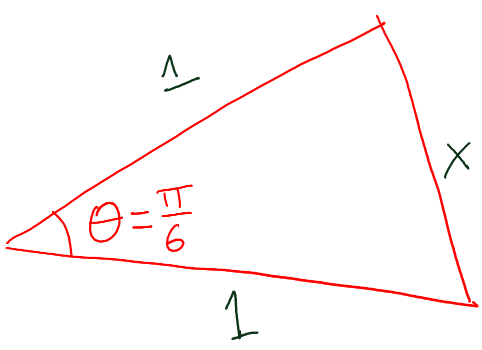
$12 \cdot \theta = 2\pi$ radians

$\theta = \frac{2\pi}{12}$ radians or $\theta = \frac{\pi}{6}$ radians.

(b) (4 points) What is the exact value of $\cos(\theta)$ (you do not need to reduce square roots.)

$\cos\left(\frac{\pi}{6}\right) = \sqrt{3}/2$

(c) (4 points) What is the exact value of the perimeter of the 12-gon (you do not need to reduce square roots.)



Law of cosines $x^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos(\theta)$

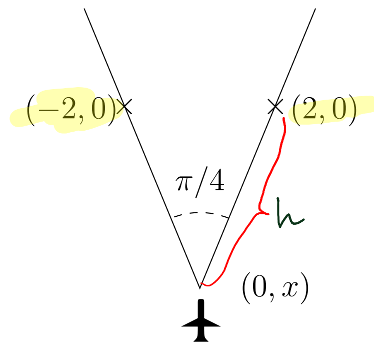
$x^2 = 2 - 2 \cdot \cos\left(\frac{\pi}{6}\right)$

$x^2 = 2 - 2 \cdot \sqrt{3}/2 = 2 - \sqrt{3}$

$x = \sqrt{2-\sqrt{3}}$.

Perimeter $= 12 \cdot \sqrt{2-\sqrt{3}}$.

2. (10 points) Two air traffic control stations are posted at coordinates of $(2,0)$ and $(-2,0)$. An airplane, at an unknown point $(0,x)$ with x negative, observe that the position between these two radio stations is $\pi/4$ radians in its field of view.



(a) (5 points) What is the distance of the plane to either of the radio stations?

If I knew the value of $\sin(\frac{\pi}{8})$ then we could compute h.

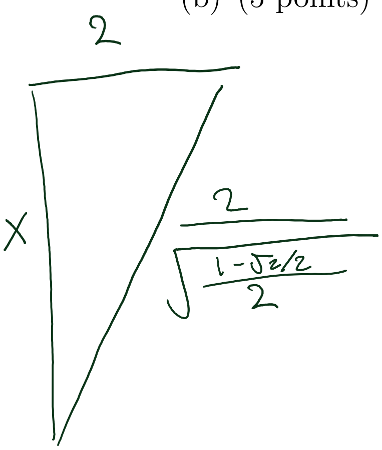
$$\sin\left(\frac{\pi}{8}\right) = \frac{2}{h} \qquad h = \frac{2}{\sin(\pi/8)}$$

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

$$\sin\left(\frac{\pi}{4}\cdot\frac{1}{2}\right) = \sqrt{\frac{1-\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}}$$

$$\rightsquigarrow \boxed{h = \frac{2}{\sqrt{\frac{1-\sqrt{2}/2}{2}}}}.$$

(b) (5 points) What is the position of the airplane?



$$\left(\frac{2}{\sqrt{\frac{1-\sqrt{2}/2}{2}}}\right)^2 - 2^2 = x^2$$

$$\left(\frac{4}{\frac{1-\sqrt{2}/2}{2}}\right) - 4 = x^2$$

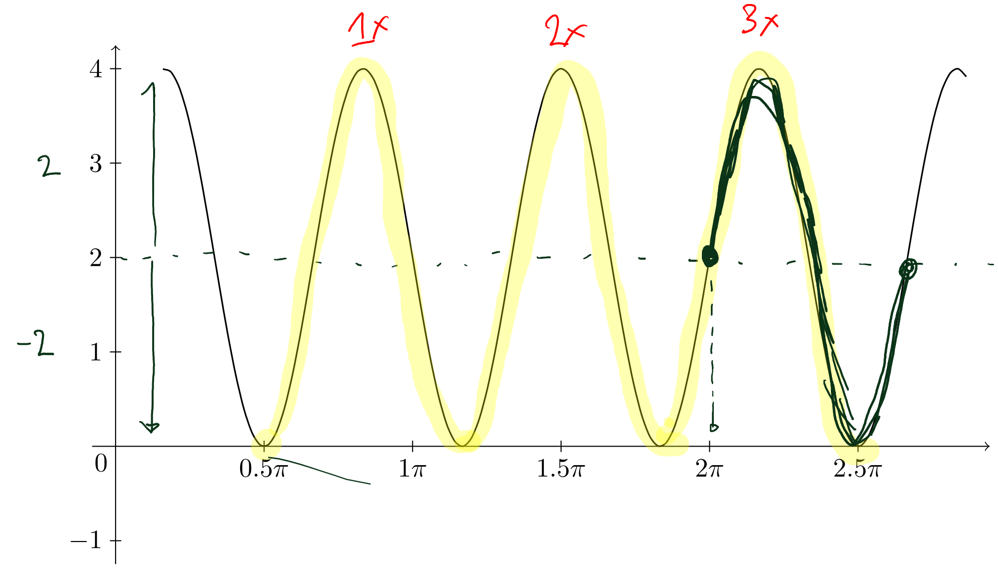
$$\frac{8}{1-\sqrt{2}/2} - 4 = x^2$$

$$\frac{16}{2-\sqrt{2}} - 4 = x^2$$

$$x = \sqrt{\frac{16}{2-\sqrt{2}} - 4}$$

$$\boxed{\text{Plane is at } \left(0, -\sqrt{\frac{16}{2-\sqrt{2}} - 4}\right)}.$$

3. (10 points) Find a function $f(x)$ whose graph is the one drawn below.



Want to determine the shifts and scalings to get this from $\sin(\theta)$.

- Amplitude (vertical scaling) = 2
- vertical shift = 2
- Period ≈ $3 \times$ period $= 2.5\pi - .5\pi$

  period $= \frac{2\pi}{3}$

  Horizontal scaling = 3 times
- Horizontal shift. 2π to the right.

$$f(x) = \underbrace{2}_{\text{vert. scaling}} \cdot \sin\Big(\underbrace{3}_{\text{horiz. scaling}} \cdot \big(x \underbrace{- 2\pi}_{\text{horiz. shift}}\big)\Big) + \underbrace{2}_{\text{vertical shift}}$$

check.

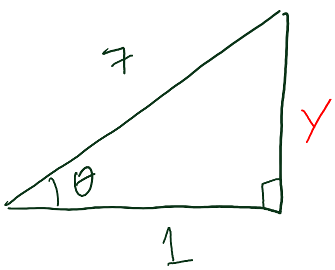
$$f(2\pi) = 2 \cdot \sin(3(2\pi - 2\pi)) + 2$$
$$= 2 \cdot \sin(3 \cdot 0) + 2$$
$$= 2 \cdot 0 + 2 = 2.$$

4. (a) (2 points) Compute $\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

In this order $\sin(\sin^{-1}(y)) = y$ always!

$$\boxed{\frac{1}{2}}$$

(b) (4 points) Compute $\sin\left(\overbrace{\cos^{-1}\left(\frac{1}{7}\right)}^{\theta}\right)$.



$\cos\theta = \frac{1}{7}$

$\theta = \cos^{-1}\left(\frac{1}{7}\right)$

$\sin(\theta) = \frac{y}{7}$

$1^2 + y^2 = 7^2$

$1 + y^2 = 49$

$y^2 = 48$

$y = \sqrt{48} = 4\sqrt{3}$

$\sin(\theta) = \frac{4\sqrt{3}}{7}$

(c) (4 points) Compute $\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right)$

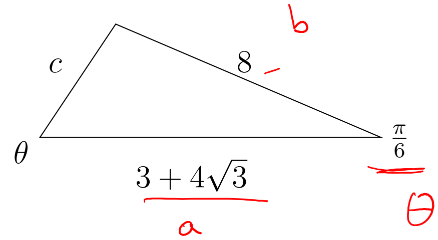
$\sin^{-1}\sin(\theta) = \theta$ only when θ is between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$.

$$\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(2\pi + \frac{\pi}{3}\right)\right)$$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{\pi}{3}.$$

5. For the triangle drawn below...



$c^2 = a^2 + b^2 - 2ab\cos\theta$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

(a) (5 points) Exactly compute the length c

$c^2 = 8^2 + (3+4\sqrt{3})^2 - 2\cdot 8\cdot(3+4\sqrt{3})\cdot\cos\frac{\pi}{6}$

$c^2 = 64 + 9 + 24\sqrt{3} + 16\cdot 3 - 16\cdot(3+4\sqrt{3})\cdot\frac{\sqrt{3}}{2}$

$= 73 + 24\sqrt{3} + 48 - 48\cdot\frac{\sqrt{3}}{2} - 16\cdot 4\sqrt{3}\cdot\frac{\sqrt{3}}{2}$

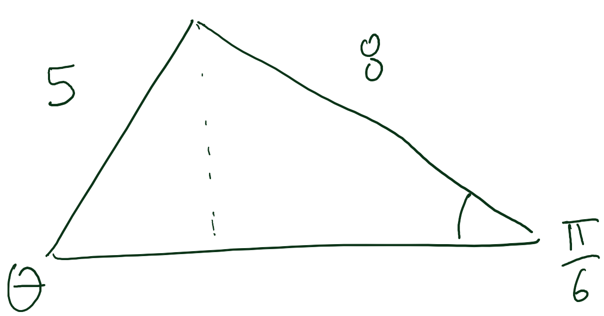
$= 121 + 24\sqrt{3} - 24\sqrt{3} - 32\cdot 3$

$= 121 - 96 = 25.$

$c^2 = 25 \qquad c = 5$

$$\begin{array}{r} \overset{1}{121} \\ -\ 96 \\ \hline 25 \end{array}$$

(b) (5 points) Exactly compute $\sin(\theta)$?

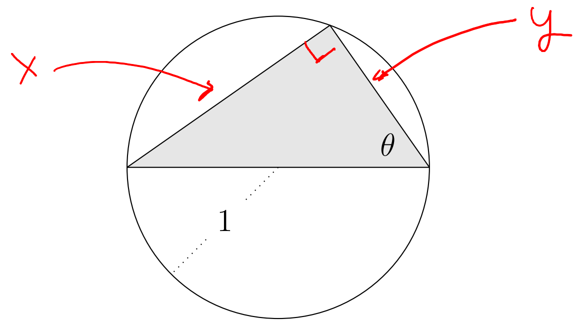


$\frac{\sin(\frac{\pi}{6})}{5} = \frac{\sin(\theta)}{8}$

$\frac{\frac{1}{2}}{5} = \frac{\sin(\theta)}{8}$

$\sin(\theta) = 4/5.$

6. Consider a right triangle whose hypotenuse is the diameter of the unit circle drawn below, and whose corner measures θ radians.



(a) (7 points) Write down the area of the triangle in terms of $\sin(2\theta)$. $= 2\cdot\sin\theta\cos\theta$.

$$\text{Area} = \frac{x\cdot y}{2}$$

$$x = 2\cdot\sin\theta$$

$$y = 2\cdot\cos\theta$$

$$\text{Area} = \frac{1}{2}\cdot a\cdot b = \frac{1}{2}(2\sin\theta\, 2\sin\theta)$$

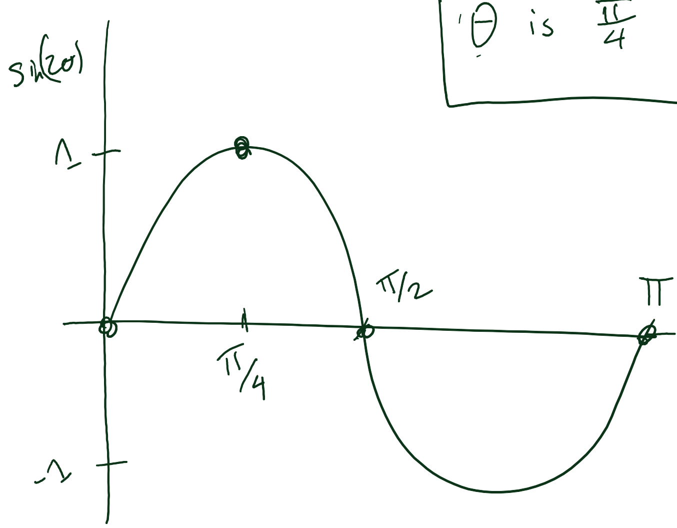
$$\sin(2\theta) = 2\sin\theta\cos\theta$$

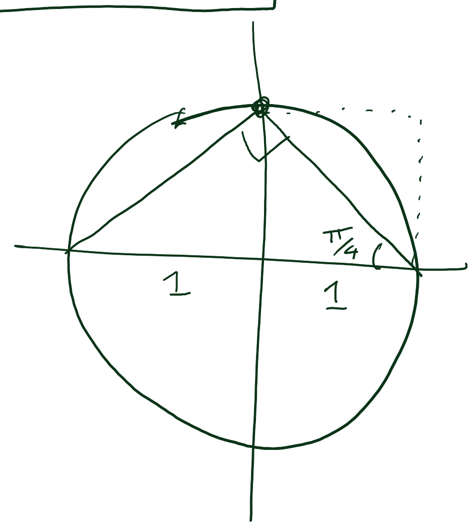
$$\text{Area} = \sin(2\theta)$$

(b) (3 points) For what value of θ does this triangle achieve its largest area?

For what θ is $\sin(2\theta)$ Largest?

$\boxed{\theta \text{ is } \frac{\pi}{4} \text{ then } \sin(2\theta) = 1.}$





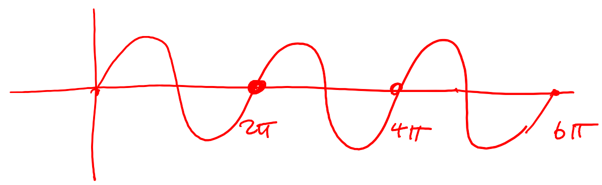
7. For each of the following statements, either mark them as true, or provide an example of why the statement is false.

(a) (1 point) If $\sin(a) = 0$, then $a = 0$ or $a = \pi$.

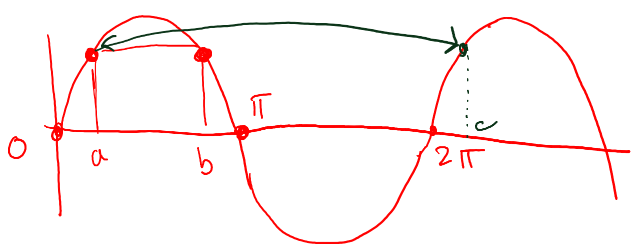
Note: if $a = 2\pi$, then $\sin(a) = 0$, so statement is false.

(b) (2 points) $\sin(a) = \sin(a + 2\pi)$.

True.



(c) (2 points) If $\sin(a) = \sin(b)$, then the difference between a and b is a multiple of 2π.

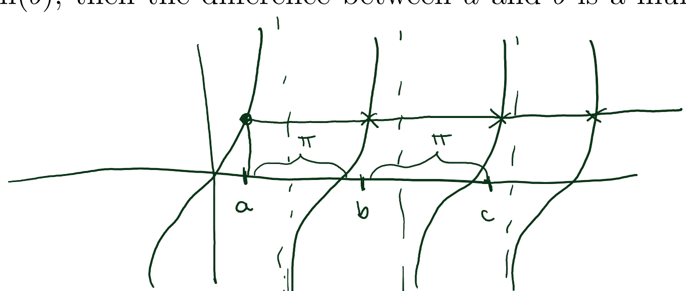


$\sin(a) = \sin(b)$

False

(d) (2 points) If $\tan(a) = \tan(b)$, then the difference between a and b is a multiple of π.

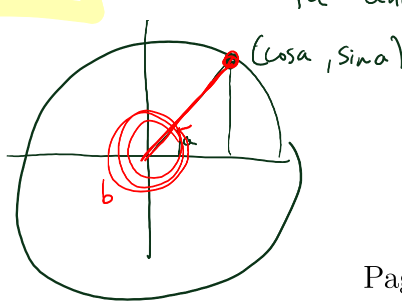
True.



(e) (3 points) If $\sin(a) = \sin(b)$ and $\cos(a) = \cos(b)$, then the difference between a and b is a multiple of 2π.

True.

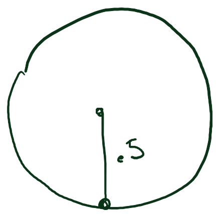
sin(a) and cos(a) determine the angle a on the unit circle



sin(b) and cos(b) is determing b on the unit circle as well.

If those 2 angles correspond to same point on unit circle, must differ by 2π multiple.

8. (a) A car which has a wheel of radius of 0.5m travels down the road. The wheel turns 6300 radians as it travels. How far did the car move?



Distance travelled = arclength of boundry swept out by angle traveled.

$= r \cdot \theta$

$= .5 \cdot 6300$ radians

$= 3150.$ meters.

(b) About how many times did the wheel turn a full rotation (round to the nearest 10).

2π radians is a full rotations

$$6300 \text{ radians} \cdot \frac{1 \text{ rotation}}{2\pi \text{ radians}}$$

$$2\pi \approx 3.14 \cdot 2 = 6.28 \approx 6.3$$

$$6300 \cdot \frac{1}{6.3} = \frac{6300}{6.3} = 1000 \text{ rotations.}$$

9. Approximate the following values:

(a) (1 point) $2e^{0.01}$.

$e^x \approx 1+x$ for x small.

$2 \cdot e^{0.01} \approx 2 \cdot (1+0.01) = 2 \cdot (1.01) = 2.02.$

(b) (2 points) $\cos(0.02)$

$\cos(x) \approx 1 - x^2$ for x small.

$\cos(0.02) \approx 1 - (.02)^2$

$= 1 - .0004 = .9996.$

$$\begin{array}{r} .02 \\ \times\ .02 \\ \hline 04 \\ 00 \\ \hline .0004 \end{array}$$

(c) (3 points) $\ln\left(\frac{0.98}{e}\right)$

$\ln(1+x) \approx x$

$\ln\left(\frac{.98}{e}\right) = \ln(.98) - \ln(e) = \ln(1 + (-.02)) - 1$

$= -.02 - 1 = -1.02.$

(d) (4 points) $\sin(3\pi + 0.01)$.

$\sin(x) \approx x$ for x Small.

$\sin(3\pi + .01) = \sin(2\pi + \pi + .01)$

$= \sin(\pi + .01)$

$\doteq -\sin(.01)$

$\approx -.01.$

10. Let $f(x) = \ln(x)$ and let $g(x) = e^{2x+1}$.

(a) (2 points) What is $(f \circ g)(0)$?

$g(0) = e^{2\cdot 0+1} = e.$

$f(g(0)) = f(e) = \ln(e) = 1.$

(b) (3 points) For which value of x is $(f \circ g)(x) = 0$?

$(f\circ g)(x) = \ln(e^{2x+1}) = 2x+1.$

$2x+1=0$

$2x=-1$

$\boxed{x=-\frac{1}{2}}$

(c) (3 points) The composition $(f \circ g)(x)$ is a line. What is the slope of this line?

$y = 2x+1$

slope of line is 2.

(d) (2 points) Write down the equation for the line $y = (f \circ g)(x)$.

$y = 2x+1$

11. (10 points) Rewrite the function

$$\frac{x^3 + x + 1}{x + 2}$$

as a polynomial plus a rational function whose numerator has smaller degree than its denominator.

$$\frac{x^3 + x + 1}{x+2} = \frac{x^3}{x+2} + \frac{x+1}{x+2}.$$

$$= \frac{x^3 + 2x^2}{x+2} + \frac{-2x^2 + x + 1}{x+2}$$

$$= x^2 + \frac{-2x^2 + x + 1}{x+2}$$

$$= x^2 + \frac{-2x^2}{x+2} + \frac{x+1}{x+2}$$

$$= x^2 + \frac{-2x^2 - 4x}{x+2} + \frac{4x + x + 1}{x+2}$$

$$= x^2 - 2x + \frac{5x+1}{x+2}$$

$$= x^2 - 2x + \frac{5x}{x+2} + \frac{1}{x+2}$$

$$= x^2 - 2x + \frac{5x+10}{x+2} + \frac{-10+1}{x+2}$$

$$= x^2 - 2x + 5 - \frac{9}{x+2}.$$

12. The biological half-life of caffeine in the human body is 5 hours, meaning that it takes roughly 5 hours for the body to break down half of the caffeine which is contained within its system.

   (a) (5 points) The typical cup of coffee contains 100 mg of caffeine. Write an equation $f(t)$ which describes the amount of caffeine in the human body after t hours from consumption of a cup of coffee.

$$f(t) = \underbrace{100\,\text{mg}}_{\text{initial amount}} \cdot \underbrace{\left(\frac{1}{2}\right)}_{\text{halving}}^{\boxed{\frac{t}{5}}}$$

\# of 5 hour periods which have occured.

   (b) (2 points) How long would one have to wait until the amount of caffeine remaining in the body is equal to 25mg?

$$100\text{mg} \xrightarrow{5\text{hrs}} 50\text{mg} \xrightarrow{5\text{hrs}} 25\text{mgs}$$

10 hours.

   (c) (3 points) At noon, Alice drinks a cup of coffee, and Bob drinks a cup of coke (which contains 25 mg of caffeine.) Let $A(t)$ describe the amount of caffeine in Alice's body after t hours, and let $B(t)$ describe the amount of caffeine in Bob's body after t hours. Compute the ratio of caffeine in their body remaining after 25 hours,

$$\frac{A(25)}{B(25)}.$$

time	0	5	10	15	
A	100	50	25	12.5	
B	25	12.5	6.25		
$\frac{A}{B}$	4	4	4	4	

$$\frac{A(t)}{B(t)} = 4$$

$$\frac{100\cdot\cancel{\left(\frac{1}{2}\right)^{\frac{t}{5}}}}{25\cdot\cancel{\left(\frac{1}{2}\right)^{\frac{t}{5}}}} = 4.$$

13. Let $f(x) = \frac{x-2}{x}$.

(a) (2 points) What is the domain of $f(x)$?

Domain is all x except 0.

$$(-\infty, 0) \cup (0, \infty)$$

parens, non [ ].

(b) (3 points) What are the horizontal asymptotes of $f(x)$?

To find horiz. asymptotes, we see what happens as $x \to \infty$, $x \to -\infty$.

$$\frac{x-2}{x} \approx \frac{x}{x} = 1.$$

$y = 1$ as $x \to \infty$

$y = 1$ as $x \to -\infty$.

(c) (5 points) Find all values of x so that $f(x) \geq 1$. Express your solution in interval notation.

$x = 1$.

$f(1) = \frac{1-2}{1} = -1$

$$\frac{x-2}{x} \geq 1$$

$x > 0$

$x - 2 \geq x$

$-2 \geq 0$

Not true, so this case cannot happen!

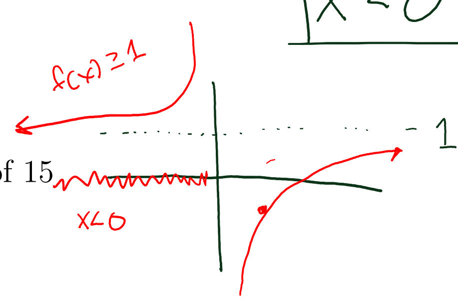
$x < 0$

$x - 2 \leq x$

$-2 \leq 0$

Always true!

$$\boxed{x < 0}.$$



14. (10 points) Find a polynomial $p(x)$ of degree 3 with the properties that

$$\left.\begin{aligned} p(-1) &= 0 \\ p(1) &= 0 \\ p(2) &= 0 \end{aligned}\right\} \text{Zeros}$$
$$p(0) = \underline{4}.$$

<u>Fact</u> if p is a polynomial, and $p(a) = 0$.

then $(x-a)$ is a factor of p.

$$p(x) = (x-a)\cdot q(x)$$

Lower degree.

$$p(x) = c\cdot(x-2)(x-1)(x+1)$$
$$= c\cdot(x-2)(x^2-1)$$
$$= c\cdot(x^3 - x - 2x^2 + 2)$$
$$= c\cdot(x^3 - 2x^2 - x + 2)$$
$$p(0) = c\cdot(0^3 - 2\cdot 0^2 - 0 + 2)$$
$$= c\cdot 2 = \underline{4}$$

$c = 2.$

$p(x) = 2x^2 - 4x^2 - 2x + 4.$