Notes on the Exponential Function

Jeff Hicks

3/13/2019

The textbook doesn't contain notes on the exponential function and compounded interest.

A first stab at compounding Interest

We start by putting P dollars in our bank account. Suppose that we have a bank account which earns a rate of r each year. Suppose that this account is compounded m times yearly. Over one m-th of a year, the amount of interest earned will be $\frac{r}{m}$. Then after a period of t years, we will earn interest $m \cdot t$ times. This means that the amount of money that we will earn is

$$\begin{aligned} P_0 \cdot & \underbrace{\left(1+\frac{r}{m}\right) \cdot \left(1+\frac{r}{m}\right) \cdots \left(1+\frac{r}{m}\right)}_{m \cdot t \text{ times}} \\ &= P_0 \left(1+\frac{r}{m}\right)^{m \cdot t} \end{aligned}$$

Now, we will ask a slightly easier question: after t years, how much will we make on an investment of one dollar at 100% interest rate.

$$\left(1+\frac{1}{m}\right)^{m \cdot t}$$

This is slightly difficult to compute – because it is an exponential function. However, we make this a *polynomial* by making the substitution $m = \frac{n}{t}$.

$$\left(1+\frac{t}{n}\right)^{n}.$$

Ok, we now have a more reasonable problem. Let's look at what this polynomial is for various quantities of n.

$$\begin{aligned} n &= 1 & \left(1+\frac{t}{1}\right)^{1} &= 1+t \\ n &= 2 & \left(1+\frac{t}{2}\right)^{2} &= 1+t+\frac{t^2}{4} \\ n &= 3 & \left(1+\frac{t}{3}\right)^{3} &= 1+t+\frac{t^2}{3}+\frac{t^3}{9} \end{aligned}$$

Ok, so we see that all of these are bigger than the linear function. But we'll have to work a bit harder to understand this example. One simplification which can be made which will give us a better idea of what is going on it to substitute $t = 1$.

n	$1+\frac{1}{n}$	$\left(1+\frac{1}{n}\right)^n$
1	2	2
2	1.5	2.25
3	1.333	2.370
4	1.25	2.441
5	1.2	2.48832
6	1.166	2.5216
100	1.01	2.7048
1000	1.001	2.7169

This seems to be getting very close to a number, which is the answer to the following question:

Question 1. *How much money do we earn with 100 percent interest after 1 year on a one dollar investment, after we compound it many, many times?*

We will now take a bit of a diversion to see why $\left(1+\frac{1}{n}\right)^n$ should be close to anything.

A return to Polynomials, and the Binomial Coefficient Theorem

We will now look at a simpler question. What is

$$(1+x)^n$$

for large values of n? Let's see what this is for some small values of n.

$$\begin{array}{llllllll}
n=1 & (1+x) & 1 & +x & & & \\
n=2 & (1+x)^2 & 1 & +2x & +x^2 & & \\
n=3 & (1+x)^3 & 1 & +3x & +3x^2 & +x^3 & \\
n=4 & (1+x)^4 & 1 & +4x & +6x^2 & +4x^3 & +x^4 \\
n=5 & (1+x)^5 & 1 & +5x & +10x^2 & +5x^3 & +x^4
\end{array}$$

Looking at the just the coefficients, we get the following numbers, which are called *Pascal's triangle*

$$\begin{array}{llllllll}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}$$

Notice that each number in this triangle is the sum of the two numbers above it and to the left. We see why this phenomenon occurs by carefully computing the $(1+x)^n = (1+x)(1+x)^{n-1}$. Let's look at why this occurs when $n=4$.

$$\begin{aligned}
(1+x)^4 =& (1+x)\cdot(1+x)^3 \\
=& (1+x)\cdot(1+3x+3x^2+1) \\
=& 1\cdot(1+3x+3x^2+1) + x\cdot(1+3x+3x^2+1) \\
=& \begin{pmatrix} 1 & +3x & +3x^2 & +x^3 & \\ & +x & +3x^2 & +3x^3 & +x^4 \end{pmatrix} \\
=& 1+(1+3)x+(3+3)x^2+(3+1)x^3+(1)x^4 \\
=& 1+4x+6x^2+4x^3+x^4
\end{aligned}$$

There is concise description the numbers which show up in the Pascals triangle.

Definition 1. *The* factorial *of a positive whole number n is the product of all the positive whole numbers smaller than n,*

$$n! := 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

For example,

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$

The factorial function grows faster than the exponential of any number!

Claim 1. *The coefficient in front of x^k in the expansion of $(1+x)^n$ is*

$$\frac{n!}{k! \cdot (n-k)!}$$

Let's look at the example of $n = 5$.

k	$\frac{n!}{k! \cdot (n-k)!}$
0	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1)(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)} = 1$
1	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1)(1 \cdot 2 \cdot 3 \cdot 4)} = 5$
2	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2)(1 \cdot 2 \cdot 3 \cdot 4)} = 10$
3	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2 \cdot 3)(1 \cdot 2)} = 10$
4	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1 \cdot 2 \cdot 3 \cdot 4)(1)} = 5$
5	$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(1)(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)} = 1$

A return to Exponential Functions

So, let's return to our expansion of

$$\left(1 + \frac{t}{n}\right)^n$$

The kth term of this expansion has a coefficient

$$\begin{aligned} \frac{n!}{k! \cdot (n-k)!} \cdot \frac{1}{n^k} &= \frac{1 \cdot 2 \cdot 3 \cdots (n-k) \cdot (n-k+1) \cdots n}{(1 \cdot 2 \cdot 3 \cdots (n-k)) \cdot (n \cdot n \cdots n \cdot n) \cdot (1 \cdot 2 \cdot 3 \cdots k)} \\ &= \frac{(n-k+1) \cdots n}{(n \cdot n \cdots n \cdot n) \cdot (1 \cdot 2 \cdot 3 \cdots k)} \end{aligned}$$

When n is very large compared to k, $(n-k+1)/n$ is very close to 1.

$$\begin{aligned} &\cong \frac{1}{1 \cdot 2 \cdot 3 \cdots k} \\ &= \frac{1}{k!} \end{aligned}$$

This means that for the smaller terms, we have an approximation:

$$\left(1 + \frac{t}{n}\right)^n \cong 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

Let's look at a couple of approximations:

k	$k!$	$\frac{1}{k!}$	Sum of first k terms
1	1	1	1
1	1	1	2
2	2	0.5	2.5
3	6	0.166666666666667	2.66666666666667
4	24	0.0416666666666667	2.70833333333333
5	120	0.00833333333333333	2.71666666666667
6	720	0.00138888888888889	2.71805555555556
7	5040	0.000198412698412698	2.71825396825397
8	40320	0.0000248015873015873	2.71827876984127
9	362880	0.00000275573192239859	2.71828152557319
10	3628800	0.000000275573192239859	2.71828180114638

As one can see, this becomes very close to a number very quickly! We call this number e.