Half Angle and Angle Sum Identities

Jeff Hicks

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UC Berkeley

Outline

- Recap: Half angle formula
- Angle sum formula

Half Angle Formulas

Computing $\sin(\theta/2)$

$\cos(2\alpha) = 1 - 2(\sin^2(\alpha))$

Let's compute $\sin(\alpha)$ from $\cos(2\alpha)$

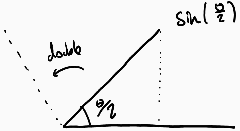
$$\cos(2\alpha) - 1 = -2(\sin(\alpha))^2$$

$$\frac{1-\cos(2\alpha)}{2} = (\sin(\alpha))^2$$

$$\pm\sqrt{\frac{1-\cos(2\alpha)}{2}} = \sin(\alpha)$$

Usually $\alpha = \frac{\theta}{2}$

$$\pm\sqrt{\frac{1-\cos(\theta)}{2}} = \sin\left(\frac{\theta}{2}\right)$$



Identity

$$\sin(\theta/2) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

Example

Find an exact formula for $\sin(\pi/12)$.

Idea: We know $\sin\left(\frac{\pi}{6}\right)$

Observe that $\frac{\pi}{12} = \frac{1}{2}\cdot\frac{\pi}{6}$

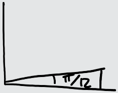
$\frac{1}{2} = \sin\left(\frac{\pi}{6}\right).$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$$\sin\left(\frac{1}{2}\cdot\frac{\pi}{6}\right) = \pm\sqrt{\frac{1-\cos(\pi/6)}{2}}$$

$$= \pm\sqrt{\frac{1-\sqrt{3}/2}{2}}$$

$$= \sqrt{\frac{1}{2} - \sqrt{3}/4}\,.$$



$\sin(\pi/12)$

Computing $\cos(\theta/2)$

$$\cos(2\alpha) = 2(\cos(\alpha))^2 - 1$$

$$1 + \cos(2\alpha) = 2\left(\cos(\alpha)\right)^2$$

$$\frac{1 + \cos(2\alpha)}{2} = \left(\cos(\alpha)\right)^2$$

$$\pm\sqrt{\frac{1 + \cos(2\alpha)}{2}} = \cos(\alpha)$$

$$\boxed{\pm\sqrt{\frac{1 - \cos(\theta)}{2}} = \cos(\theta)}$$

$$\alpha = \frac{\theta}{2}$$

Computing $\tan(\theta/2)$

$$\tan(\alpha) = \cancel{\frac{\sin(\theta)}{\cos(\theta)}} = \cancel{\frac{2\sin(\theta)\cos(\theta)}{2\cos^2(\theta)}}$$

2x angle form for sine

$$\frac{\sin(\alpha)}{\cos(\alpha)} = \frac{2\sin(\alpha)\cos(\alpha)}{2\cdot\cos^2(\alpha)} = \frac{\sin(2\alpha)}{2\cos^2(\alpha)}.$$

$$= \frac{\sin(2\alpha)}{2\cos^2(\alpha) - 1 + 1}$$

2x angle for cosine.

$$= \frac{\sin(2\alpha)}{\cos(2\alpha) + 1}$$

$$\alpha = \frac{\theta}{2}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{\cos(\theta)+1}.$$

Half Angle Formulas

Identity

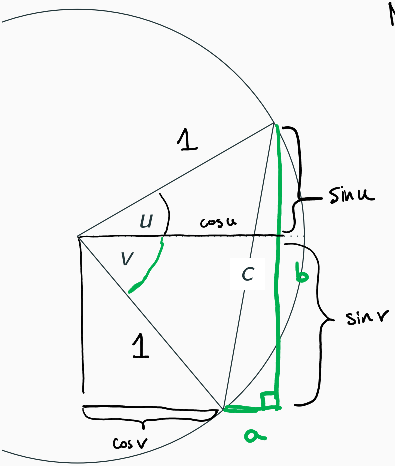
$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$$

$$\overset{\text{sin}}{\cancel{\cos}}\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1+\cos(\theta)}$$

Angle Sum Formula

Computing $\cos(u+v)$.



Now we'll compute the length c in 2 ways

$$c^2 = 1^2 + 1^2 - 2\cos(u+v)$$

$$b = \sin(u) + \sin(v)$$

$$a = \cos(u) - \cos(v).$$

$$c^2 = (\sin(u) + \sin(v))^2 + (\cos(u) - \cos(v))^2$$

$$1 + 1 - 2\cos(u+v) = \sin^2 u + 2\sin u \sin v + \sin^2 v + \cos^2 u - 2\cos u \cos v + \cos^2 v.$$

$$1 + 1 - 2\cos(u+v) = 1 + 2\sin u \sin v - 2\cos u \cos v + 1$$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v.$$

Angle sum for Cosine

Identity

$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v).$

Compute this when $u = 0$.

LHS $\cos(0+v) = \cos(v)$

RHS $\cos(0)\cos(v) - \sin(0)\sin(V)$

$1 \cos(v) - 0 \cdot \sin(V)$

$= \cos(V)$.

Application: Right angle identity for cosine

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$$\cos\left(u+\frac{\pi}{2}\right) = \cos(u)\cdot\cos(\pi/2) - \sin(u)\cdot\sin(\pi/2)$$

$$= \cos(u)\cdot 0 \quad - \sin(u)\cdot 1$$

$$= -\sin(u)$$

Conclusion: $\cos\left(u+\frac{\pi}{2}\right) = -\sin(u)$.

Example: Exactly compute $\cos(\overset{7}{\cancel{5}}\pi/12)$

Note: $\dfrac{7\pi}{12} = \dfrac{3\pi}{12} + \dfrac{4\pi}{12}$

$$= \frac{\pi}{4} + \frac{\pi}{3}$$

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\cdot\sin\frac{\pi}{3}$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$$

$$= \sqrt{2}/4 - \sqrt{6}/4 \qquad \square.$$

Example: $\cos\left(\frac{3\pi}{64}\right) \rightsquigarrow$ Half angle to get $\cos\left(\frac{\pi}{64}\right)$

$$\cos\left(\frac{\pi}{64} + \frac{\pi}{64} + \frac{\pi}{64}\right)$$

Angle sum for Sine

Recall that $\sin(\theta) = \cos(\theta - \pi/2)$

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$v = w - \frac{\pi}{2}$$

$$= \cos\left(u + w - \frac{\pi}{2}\right) = \cos u \cos\left(w - \frac{\pi}{2}\right) - \sin(u) \cdot \sin\left(w - \frac{\pi}{2}\right)$$

$$= \sin(u+w) = \cos u \cdot \sin w + \sin(u) \cos(w)$$

Angle sum for Sine

Identity

$\sin(u+v) = \cos(u)\sin(v) + \sin(u)\cos(v)$

Computing $\tan(u+v)$

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)}$$

$$= \frac{\sin(u)\cos(v) + \sin(v)\cos(u)}{\cos(u)\cos(v) - \sin(u)\sin(v)}.$$

Divide both num & denominator by $\cos u \cos v$.

$$= \frac{\dfrac{\sin u \cos v}{\cos u \cos v} + \dfrac{\sin(v)\cos u}{\cos(u)\cos(v)}}{\dfrac{\cos(u)\cos(v)}{\cos(u)\cos(v)} - \dfrac{\sin(u)\sin(v)}{\cos(u)\cos(v)}}.$$

$$= \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

Angle sum for tangent

Identity

$\tan(u+v) = \frac{\tan(u)+\tan(v)}{1-\tan(u)\tan(v)}$.

Exactly compute $\tan(7\pi/12)$

$$\tan(7\pi/12) = \tan(\pi/4 + \pi/3)$$

$$= \frac{\tan \pi/4 + \tan(\pi/3)}{1 - (\tan \pi/4)\cdot(\tan \pi/3)}$$

$$= \frac{1 + \sqrt{3}}{1 - 1\cdot\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}}.$$

Angle difference formulas

$$\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$$
$$\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$$
$$\tan(u+v) = \frac{\tan(u)+\tan(v)}{1-\tan(u)\tan(v)}.$$

Angle sum formulas

$\sin(u-v) = \sin u \cos v - \cos(u) \sin(v)$

Application: Beats between two frequencies

Sum of Cosine Waves

$$\cos(u+v)\cos(u-v)$$

Sound and sin waves.

Sounds

$$\text{pressure against ear } (t) = \sin\left(440 \cdot 2\pi \cdot t \right)$$

Temporal Aliasing

From previous slide, we have the identity
$\cos(u+v)\cos(u-v) = \frac{1}{2}\cos(2u) + \cos(2v)$